CS 395/495-26: Spring 2002

IBMR: Week 6B

Chapter 3: Estimation & Accuracy

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Reminders

No midterm, no final, but …
Alternating homework / project

- Project 2 Due Thurs May 9
- Homework 1 due Thurs May 16

- New! Project 3 assigned May 9 (see website)
  due May 23

R^4 Quadrics for the Variety $\mathcal{V}_H$

- For the i-th point pair, DLT minimizes ϵ_i:

$A_i h = \epsilon_i$

$$\begin{bmatrix} 0^T & -w'_i \mathbf{x}_i^T & y'_i \mathbf{x}_i^T \\ w'_i \mathbf{x}_i^T & 0^T & -x'_i \mathbf{x}_i^T \\ -y'_i \mathbf{x}_i^T & -x'_i \mathbf{x}_i^T & 0^T \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix}$$

- In R^4, point pair is $X_i = [x\ y\ x'\ y']^T$ (assume w=w'=1)
- Augment X_i with 1; then R^4 quadrics are:

$\boxed{X^T Q X = 0}$ or $[x\ y\ x'\ y'\ 1] \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & Q & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ y \\ x' \\ y' \\ 1 \end{bmatrix} = 0$

R^4 Quadrics for the Variety $\mathcal{V}_H$

- For the i-th point pair, DLT minimizes $\boldsymbol{\epsilon}_i$:

$A_i \cdot h = \boldsymbol{\epsilon}_i$

$$\begin{array}{l} \text{Row 1} \rightarrow \\ \text{Row 2} \rightarrow \\ \text{Row 3} \rightarrow \end{array} \begin{bmatrix} 0^T & -w'_i \mathbf{x}_i^T & y'_i \mathbf{x}_i^T \\ w'_i \mathbf{x}_i^T & 0^T & -x'_i \mathbf{x}_i^T \\ -y'_i \mathbf{x}_i^T & -x'_i \mathbf{x}_i^T & 0^T \end{bmatrix} \begin{bmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{bmatrix} = \begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix}$$

- Write each row as an R^4 quadric: $\boxed{X^T Q_i X = 0}$

Q_{i1} for Row 1 | Q_{i2} for Row 2 | Q_{i3} for Row 3

$$\begin{bmatrix} 0 & 0 & 0 & h_{31} & -w'h_{21} \\ 0 & 0 & 0 & h_{32} & -w'h_{22} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & wh_{22} \\ 0 & 0 & 0 & 0 & w'wh_{33} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -h_{31} & 0 & w'h_{11} \\ 0 & 0 & -h_{32} & 0 & w'h_{12} \\ 0 & 0 & 0 & 0 & -w\,h_{33} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w'wh_{13} \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & h_{21} & -h_{11} & 0 \\ 0 & 0 & h_{22} & -h_{12} & 0 \\ 0 & 0 & 0 & 0 & w\,h_{23} \\ 0 & 0 & 0 & 0 & w\,h_{13} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R^4 Quadrics for the Variety $\mathcal{V}_H$

- Write each row as an R^4 quadric: $\boxed{X^T Q_i X = 0}$
- ALL have this form: $Q_i = \begin{bmatrix} 0 & 0 & a & b & c \\ 0 & 0 & d & e & f \\ 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & j \end{bmatrix}$

(Can also write as symmetric without changing meaning!)

- Quadric's derivatives are easy:

$$\frac{\partial Q_i}{\partial x}: \begin{array}{c|c} \begin{array}{c} x \\ y \\ x' \\ y' \\ 1 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & c \end{bmatrix} \\ & \begin{array}{ccccc} x & y & x' & y' & 1 \end{array} \end{array} \quad \frac{\partial Q_i}{\partial y}: \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & f \end{bmatrix} \\ \begin{array}{ccccc} x & y & x' & y' & 1 \end{array} \end{array} \quad \frac{\partial Q_i}{\partial x'}: \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g \end{bmatrix} \\ \begin{array}{ccccc} x & y & x' & y' & 1 \end{array} \end{array} \quad \frac{\partial Q_i}{\partial y'}: \begin{array}{c|c} \begin{bmatrix} 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h \end{bmatrix} & \begin{array}{c} x \\ y \\ x' \\ y' \\ 1 \end{array} \\ \begin{array}{ccccc} x & y & x' & y' & 1 \end{array} & \end{array}$$

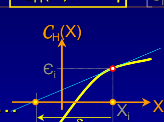
Sampson Error Approx.

- Sampson error's estimate: $\hat{X}_i = X_i + \delta_X$
- Re-name DLT result $A_i \cdot h = \boldsymbol{\epsilon}_i$ as $\boxed{\mathcal{C}_H(X) = \boldsymbol{\epsilon}_i} = \begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix}$

(Note: $\mathcal{C}_H(X)$ is a stack of 2(or 3) rows of A)

- Use Taylor Series Approx:

$$\mathcal{C}_H(X_i + \delta_X) \cong \mathcal{C}_H(X) + \frac{\partial \mathcal{C}_H(X_i)}{\partial X}\delta_X + \ldots$$



- Find δ_x for error $\cong 0$:

$$\mathcal{C}_H(X_i + \delta_X) \cong \mathbf{0} = \boldsymbol{\epsilon}_i + \frac{\partial \mathcal{C}_H(X_i)}{\partial X}\delta_X$$

- But How? (2-,3- and 4-vectors here!)

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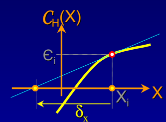
- Sampson error's estimate: $\hat{X}_i = X_i + \delta_X$
- Re-name DLT result $A_i h = \epsilon_i$ as $\boxed{\mathcal{C}_H(X) = \epsilon_i} = \begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix}$
- Find δ_x: solve $\epsilon_i + \frac{\partial \mathcal{C}_H(X_i)}{\partial X} \delta_X = 0$
- Or as in Book: $\epsilon_i + \mathbf{J}\,\delta_X = 0$ (pg. 82)



$$\begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial y'} \\ \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y'} \\ \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y'} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix} = \mathbf{0}$$

Find J terms from R^4 quadrics, solve for δ_X …

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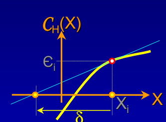

$$\begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial y'} \\ \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y'} \\ \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y'} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix} = \mathbf{0}$$

!TROUBLE! rank=2, but 4 unknowns!

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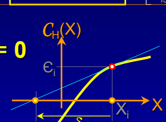
$$\begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix} + \begin{bmatrix} \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H1}(X_i)}{\partial y'} \\ \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y'} \\ \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial x'} & \frac{\partial \mathcal{C}_{H2}(X_i)}{\partial y'} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix} = \mathbf{0}$$



- Sampson Error vector

$$\boxed{\delta_X = -J^T(JJ^T)^{-1}\epsilon}$$

Answer: find <u>shortest</u> δ_X: minimize $\|\delta_X\|^2$ by Lagrange multipliers…

Aside: Lagrange Multipliers

"Given only $f(x) = 0$, find shortest-length x"

Semi-magical vector trick:

- Length² is scalar value: $||x||^2 = x^T x$
- Mix in an unknown vector λ ('Lagrange multiplier'):

$$x^T x \; - \; 2\,\lambda^T f(x) = g(x,\lambda) = \text{weird\_length}^2$$

Aside: Lagrange Multipliers

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Length²

(MUST be = 0)

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Length²

(MUST be = 0)

Find shortest 'weird_length²':

- Find $(\partial g / \partial x) = 0$, Find $(\partial g / \partial \lambda) = 0$,
- Solve them for λ (substitutions)
- then use λ in $g(x,\lambda)$ to find x !Done!

More General Approach:

- Define 'Measurement space' (e.g. R^4)
  - Vector: a complete measurement set
  - Vector holds only known values, but
  - Can be ANYTHING: input,output, length, H, …
  - All possible measures (good or bad)
- Define 'Model'
  - All possible error-free 'perfect measurements' such as the variety $\mathcal{V}_H$
  - Defined by a known function (e.g. $\mathcal{C}_H(X) = 0$)
  - A subset of measurement space

More General Approach:

- Gather points X_i in 'measurement space'
- Replace X_i with 'estimates' $\hat{X}_i$ that:
  - Satisfy the model, and
  - minimize distance $|| \hat{X}_i - X_i ||$
- Book (pg 85) gives examples for:
  - Error in both images (what we just did)
  - Error in one image only
  - Maximum likelihood estimation:
    - → same as minimizing geometric error ←
    - → same as minimizing reprojection error ←

DLT Non-Invariance

!Accuracy depends on origin location!

- 'Non-optional' Correction required: (Outline on pg. 92)
  - Find centroid of measured points
  - Find average distance of points to centroid
  - Offset origin to centroid for all points
  - Scale about new origin to set average distance-to-origin to sqrt(2)
  - Reverse process after DLT. See book for proof.

END