CS 395/495-26: Spring 2002

IBMR: Week 5 A

Finish Chapter 2: 3D Projective Geometry + Applications

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Project 2 Hints

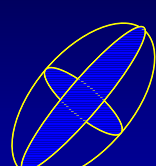
- 4 point correspondence:
  - Book shows: $x' (h_{31} x + h_{32} y + h_{33}) = (h_{11} x + h_{12} y + h_{13})$
    $y' (h_{31} x + h_{32} y + h_{33}) = (h_{11} x + h_{12} y + h_{13})$
  - Rearrange: known vector (dot) unknown vector

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = 0$$

  - stack, solve for null space…
- Hint Files
  - Added 'max' commands, examples
  - Make, test your own H matrices!

Quadrics Summary

Quadrics are the 'x^2 family' in $\mathbf{P}^3$:
  - Point Quadric: $\mathbf{x}^T \mathbf{Q} \mathbf{x} = 0$
  - Plane Quadric: $\pi^T \mathbf{Q}^* \pi = 0$
- Transformed Quadrics:
  - Point Quadric: $\mathbf{Q}' = \mathbf{H}^{-T} \mathbf{Q} \mathbf{H}$
  - Plane Quadric: $\mathbf{Q}^{*\prime} = \mathbf{H} \mathbf{Q}^* \mathbf{H}^T$

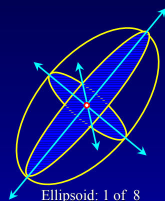


Ellipsoid: 1 of 8 quadric types

- Symmetric Q, Q* matrices:
  - **10** parameters but **9 DOF**; 9 points or planes
  - (or less if degenerate…)
  - 4x4 symmetric, so SVD(**Q**) = $\mathbf{USU}^T$

Quadrics Summary

- SVD(Q) = **USU**^{T}:
  - **U** columns are quadric's **axes**
  - **S** diagonal elements: scale
- On U axes, write any quadric as:
  $au_1^2 + bu_2^2 + cu_3^2 + d = 0$

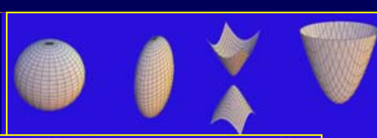


Ellipsoid: 1 of 8 quadric types

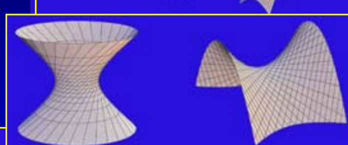
- Classify quadrics by
  - sign of a,b,c,d: (>0, 0, <0)
- Book's method:
  - scale a,b,c,d to (+1, 0, –1)
  - classify by Q's **rank** and (**a+b+c+d**)

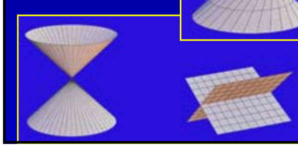
Quadrics Summary

All Unruled Quadrics are Rank 4: (See page 55)



BUT Some Rank 4 quadrics are Ruled:

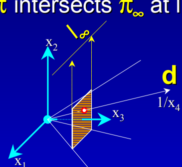




and All degenerate quadrics (Rank<4) are Ruled (or Conic)

P^3's Familiar Weirdnesses

- The plane at infinity: $\pi_\infty = (0\ \ 0\ \ 0\ \ 1)^T$
  (this is the 2D set of all…)
- 'Ideal Points' at infinity: $d = p_\infty = (x_1\ x_2\ x_3\ 0)^T$
  - (Also called 'direction' **d** in book)
- Parallel Planes intersect at a line within π_∞
- Parallel Lines intersect at a point within π_∞
- Any plane π intersects π_∞ at line l_∞ (put π in P^2)



as x4→0,
points (x_1,x_2,x_3,x_4)→ infinity

P^3's Familiar Weirdnesses

- The plane at infinity: $\pi_\infty = (0 \;\; 0 \;\; 0 \;\; 1)^T$
- Only H_p transforms π_∞ (stays π_∞ for $H_S H_A$)
  - Recall $\pi' = \mathbf{H}^{-T} \pi$
  - **Careful!** H_S, and H_A and move points within π_∞
- Both H_P and π_∞ have 3DOF
  - use one to find the other:
    - Find π_∞' in image space; use π_∞ in world-space to find $\mathbf{H_P}$
    - Find directions in π_∞ with known angles in P^3

New Weirdness: Absolute Conic Ω_∞

- **WHY learn Ω_∞?** Similar to C_∞ for P^2…
  - Angles from directions (d_1, d_2) or planes (π_1, π_2)
  - π_∞ has 3DOF for H_P; Ω_∞ has 5DOF for H_A
- Ω_∞ Requires TWO equations:

Ω_∞ : $x_1^2 + x_2^2 + x_3^2 = 0$,	or '2D point conic where C = I'
$x_4 = 0$,	or 'all points are on π_∞'

- Ω_∞ is complex 2D Point Conic on the π_∞ plane (?!?!?!)
- Recall plane at infinity $\pi_\infty = [\,0,\; 0,\; 0,\; 1]^T$ holds 'directions' $d = [x_1, x_2, x_3, 0]^T$

New Weirdness: Absolute Conic Ω_∞

- Ω_∞ is complex 2D Point Conic on the π_∞ plane

$x_1^2 + x_2^2 + x_3^2 = 0$,	or '2D point conic where C = I'
$x_4 = 0$,	or 'all points are on π_∞'

- Only $H_A H_P$ transforms Ω_∞ (stays Ω_∞ for H_S)
- All circles (in any π) intersect Ω_∞ circular pts.
  - (recall: circular pts. hold 2 axes: $x \pm iy$)
- All spheres (in P^3) intersect π_∞ at all Ω_∞ pts.

(not clear what this reveals to us)

New Weirdness: Absolute Conic Ω_∞

Ω_∞ measures angles between Directions (d_1, d_2)

- World-space Ω_∞ is I_{3x3} (ident. matrix) within π_∞
- Image-space Ω_∞' is transformed (How? as part of π_∞?)
- Euclidean world-space angle θ is given by:

$$\cos(\theta) = \frac{(d_1^T \Omega_\infty' \, d_2)}{\sqrt{(d_1^T \Omega_\infty' \, d_1)(d_2^T \Omega_\infty' \, d_2)}}$$

- Directions d_1, d_2 are orthogonal iff $d_1^T \Omega_\infty' \, d_2 = 0$
- ?!?! What is Ω_∞ in P^3? Perhaps $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$? but….

(do not change)

Absolute Dual Quadric Q^*_∞

Exact Dual to Absolute Conic Ω_∞ in plane π_∞

- Any conic in a plane = a degenerate quadric
  - (even though plane π_∞ consists of all points at infinity)
  - (even though conic Ω_∞ has no real points, all 'outer limits' of π_∞)

Q^*_∞ is a Plane Quadric that matches Ω_∞

- Defined by tangent planes π (e.g. $\pi^T Q^*_\infty \pi = 0$)
- Conic Ω_∞ is on the 'rim' of quadric Q^*_∞
- In world space, $Q^*_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Image space: **8DOF** (up to similarity) (?same as Ω_∞ ?)

Absolute Dual Quadric Q^*_∞

- In world space, $Q^*_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\pi_\infty = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
- Q^*_∞ always has infinity plane π_∞ as tangent
  $Q^*_\infty \, \pi_\infty = 0$ and $Q^*_\infty{}' \, \pi_\infty{}' = 0$
- Find angles between planes π_1, π_2 with Q^*_∞:

$$\cos(\theta) = \frac{(\pi_1^T Q^*_\infty{}' \, \pi_2)}{\sqrt{(\pi_1^T Q^*_\infty \, \pi_1)(\pi_2^T Q^*_\infty \, \pi_2)}}$$

- Can **test** for $\perp$ planes π_1, π_2 : $\pi_1^T Q^*_\infty{}' \, \pi_2 = 0$

End of Chapter 2. Now what?

Absolute Dual Quadric Q^*_∞

How can we find Q^*_∞' ?

- From ⊥ plane pairs (flatten, stack, null space…)

How can we use it?

- Transforms: "Q^*_∞ fixed iff H is a similarity" means
  Changes Q^*_∞ to Q^*_∞' unless H is a similarity
  (Recall that $Q^*_\infty{}' = H\ Q^*_\infty H^T$)
- **THUS** known Q^*_∞' can solve for $\mathbf{H_A H_P}$
  - Q^*_∞' is symmetric, so SVD()=USU^T; so by inspection: H=U

What else can we DO in P^3?

View Interpolation! Find H (or its parts: $\mathbf{H_S H_A H_P}$)

- By ⊥ Plane Pairs ($\pi_1^T Q^*_\infty \pi_2 = 0$)…
  - Find Q^*_∞ by flatten/stack/null space method
  - Find $\mathbf{H_A H_P}$ from $Q^*_\infty \rightarrow Q^*_\infty$ relation (symmetric, so use SVD)

Simpler:

- By Point (or Plane) Correspondence
  - Can find full H (15DOF) in $\mathbf{P^3}$ with 5 pts (planes)
  - Just extend the $\mathbf{P^3}$ method (see Project 2)

But what can we DO in P^3?

View Interpolation! Find H (or its parts: $\mathbf{H_S H_A H_P}$)

- By Parallel Plane pairs (they intersect at π'_∞)
  - Find π'_∞ by flatten/stack/null space method,
  - Solve for H_p using $H_P^{-T}\ \pi_\infty = \pi'_\infty$
- By ⊥ Direction Pairs ($d_1\ \Omega'_\infty\ d_2 = 0$)…
  - Find Ω'_∞ from flatten/stack/null space method,
  - Find H_A from $\Omega_\infty \rightarrow \Omega'_\infty$ relation
  - (symmetric, so use SVD…)

What can we DO in P^3?

Open questions to ponder:

- Can you do line-correspondence in P^2? in P^3?
- How would you find angle between two lines whose intersection is NOT the origin?
- Can you find **H** from known angles, $\theta \neq 90^\circ$?
- How can we adapt P^2 'vanishing point' methods to P^3?
- How might you find **H** using twisted cubics?
  using the Screw Decomposition?
- Given full 3D world-space positions for pixels ('image+depth), what **H** matrix would you use to 'move the camera to a new position'?
- What happens to the image when you change the projective transformations **H** (bottom row)?

END