CS 395/495-26: Spring 2002

IBMR: Week 4 B

Chapter 2: 3D Projective Geometry

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Projective Transformations

- Use H for transforms in P^3:
  - Has 15 DOF (4x4 -1)
  - Superset of the P^2 **H** matrix:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$$

$$H_2 = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{21} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} h_{11} & h_{12} & 0 & h_{13} \\ h_{21} & h_{22} & 0 & h_{23} \\ 0 & 0 & 0 & 0 \\ h_{31} & h_{32} & 0 & h_{33} \end{bmatrix}$$

P^3 Transformations

- Transform a point **p** or plane π with **H**:

$$p' = H \cdot p \qquad \pi' = H^{-T} \cdot \pi$$

- Lines 1: Transform a span:

$$W' = H \cdot W \qquad W^{*\prime} = H^{-T} \cdot W^*$$

- Lines 2: Transform a Plűcker Matrix:

$$L' = H \cdot L \cdot H^T \qquad L^{*\prime} = H^{-T} \cdot L^* \cdot H^{-1}$$

The bits and pieces of H_3

H_3 has 15 independent variables (DOF)

- Computer Graphics method (4x4 matrix):

3D Translation	(t_x, t_y, t_z)	
3D Scale	(s_x, s_y, s_z)	
3D Rotation	$(\theta_x, \theta_y, \theta_z)$	
3D Skew	(s_{xy}, s_{xz}, s_{yz})	(rarely nonzero)
3D Projection	(v_x, v_y, v_z)	(v_x, v_y rarely nonzero)

- Computer Vision method(3D projective):
  - Euclidean -- 6DOF(3D translate t_x, t_y, t_z; 3D rotate $\theta_x, \theta_y, \theta_z$;)
  - Similarity -- 7DOF (add uniform scale s;)
  - Affine --12DOF (add skew (3DOF), directed scale(2))
  - Projective--15DOF (changes x_4; ?4D-rotation-like?)

The bits and pieces of H_3

H_3 has 15 independent variables (DOF)

- Decomposable into 4 useful parts: (pg 59)

$$H = H_S\, H_A\, H_P = \left[\begin{array}{ccc|c} sR & \cdot & \cdot & t \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline 0^T & \cdot & \cdot & 1 \end{array}\right] \left[\begin{array}{ccc|c} K & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline 0^T & \cdot & \cdot & 1 \end{array}\right] \left[\begin{array}{ccc|c} I & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline v^T & \cdot & \cdot & v \end{array}\right]$$

- Similarity H_S:
  - 3D translate, rotate, uniform scale only (7DOF)
- Affine H_A:
  - non-uniform scale (5DOF)
- Similarity H_P:
  - Projective coupling for x_4 (3DOF)

(Can move scaling DOF to affine to get 6DOF for both H_S and H_A)

P^2 Conics → P^3 Quadrics

Recall Conics are the ‘x^2 family’ in P^2:

- Point Conic: $x^T C x = 0$
- Line Conic: $L^T C^* L = 0$

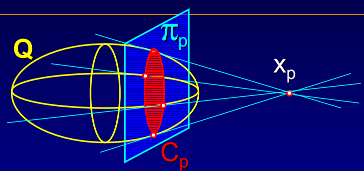
C, C^* == 3x3 matrix

Similarly, Quadrics are the ‘x^2 family’ in P^3:

- Point Quadric: $x^T Q x = 0$
- Plane Quadric: $\pi^T Q^* \pi = 0$

Q, Q^* == 4x4 matrix

Quadric Properties



- Quadric Q and point x_p Form a 'Polarity':
  - maps point←→plane, $\pi_p = Qx_p$
    (conics map point←→line)
  - Intersection of **Q** with plane π_p is a conic C_p
    (not easy to find)
  - **Q**'s tangent planes at C_p all intersect at x_p

Quadric Properties

- **Q** and **Q*** are 4x4 symmetric matrices
- 10 params, but only **9 DOF**
  - Find from 9 points or planes (not lines!)
  - Rank<3? 'Degenerate', fewer DOF
- Transformed Quadrics:
  - Point Quadric: $x^T Q x = 0$ use $Q' = H^{-T} Q H$
  - Plane Quadric: $\pi^T Q^* \pi = 0$ use $Q^{*\prime} = H Q^* H^T$

Quadric Properties

- Quadric Q is symmetric; thus SVD is too:
  SVD(**Q'**) = USU^T
- Recall:
  - Transformed Point Quadric: $Q' = H^{-T} Q H$
  - Transformed Plane Quadric: $Q^{*\prime} = H Q^* H^T$
- ? Know a quadric before & after transform?
  → SVD helps find that transform←
- SVD matrix 'S' can classify any quadric
  - No real points, Sphere/ellipsoid, Hyperboloid, one point, origin cone, one line, two planes… (see pg 55)

Quadric Properties

- SVD finds an orthornormal input basis U; in 'u' coordinates, can write any quadric as:
  $au_1^2 + bu_2^2 + cu_3^2 + d = 0$
- Classify quadrics by sign of a,b,c,d
- Rank 4 Quadrics; (nonzero a,b,c,d)
  - 'No real points': a,b,c,d > 0
  - Sphere/ellipsoid: a,b,c > 0, d<0
  - Hyperboloid: d<0, one of a,b,c <0
- Rank<4? Degenerate, Ruled Quadrics
  - one point, cone at origin, single line, two planes, one plane.

Twisted Cubics

- Recall 2D conics are the 'x^2 family':
  - Parameterize $x^T \mathbf{C}\, \mathbf{x} = \mathbf{0}$ by 't'; find $x_1(t)$, $x_2(t)$…
  - Write as matrix eqn:
  - 1D Quadratic (t^2 family)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \mathbf{A} & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$$

- Easy to extend to P^3:
  - A 1D cubic curve (t^3 family)
  - Wanders in P^3
  - Not restricted to plane

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{A} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Twisted Cubics (TC)

- Has 12 DOF (15 – 3 due to 1-D parameter)
- Specified by 6 points in P^3
  (each point constrains 2DOF)
- TC transformed by **H** → another TC

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{A} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

P^3's Familiar Weirdnesses

- The plane at infinity: $\pi_\infty = (0 \quad 0 \quad 0 \quad 1)^T$
- Ideal Points at infinity: $p_\infty = (x_1 \; x_2 \; x_3 \; 0)^T$

- Parallel Planes intersect at a line within π_∞
- Intersection of π_∞ with plane π is l_∞ (in P^2)
- Parallel Lines intersect at a point within π_∞
- π_∞ affected **ONLY** by H_p (stays for $H_S H_A$)

  Both H_P and π_∞ have 3DOF… (solve!)

P^3's Familiar Weirdnesses

- In world space, we know $\pi_\infty = (0 \quad 0 \quad 0 \quad 1)^T$
- Find π'_∞ in image space, solve for H_p:

$$\pi'_\infty = H_p^{-T} \pi_\infty$$

But How?

- **Absolute Conic: C^*_∞ AEmbedded in π_∞**
- **Absolute Dual Quadric**
- **very similar to C^*_∞ process…**

END

END2