Inference in Markov Networks

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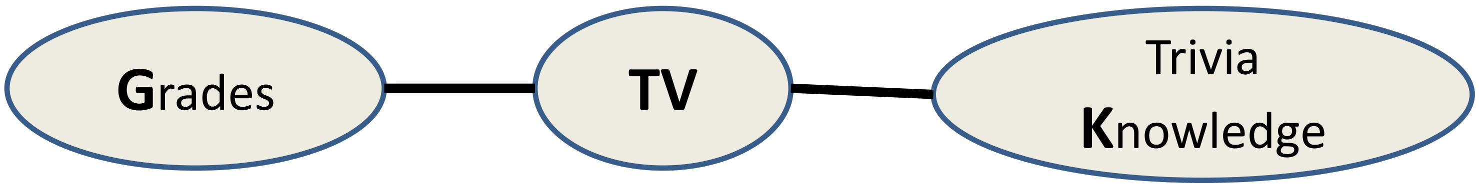
Northwestern EECS 395/495 Fall 2014

Markov Network Inference

- $P(\boldsymbol{x}) = \dfrac{\prod_c \phi_c(\boldsymbol{x}_c)}{Z}$ $\qquad Z = \sum_{\boldsymbol{x}} \prod_c \phi_c(\boldsymbol{x}_c)$

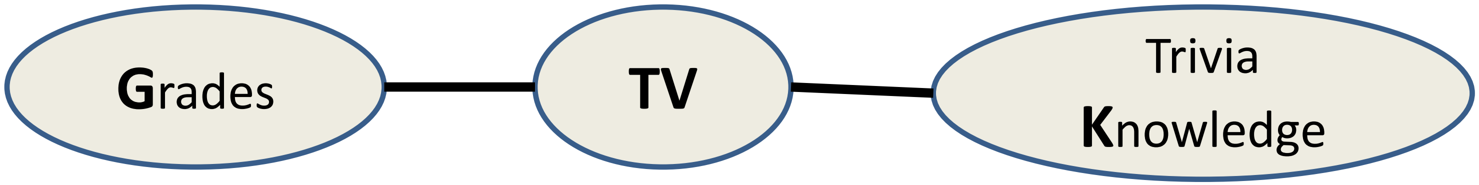


Grades	TV	ϕ_1(G, TV)
bad	none	2.0
good	none	3.0
bad	lots	3.0
good	lots	1.0

TV	Trivia Knowledge	ϕ_2(TV, K)
none	weak	2.0
lots	weak	1.0
none	strong	1.5
lots	strong	3.0

Markov Network Inference

- P(Grades | TV=none)? Straightforward: enumerate, then re-normalize



Grades	TV	ϕ_1(G, TV)
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TV	Trivia Knowledge	ϕ_2(TV, K)
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none	strong	1.5
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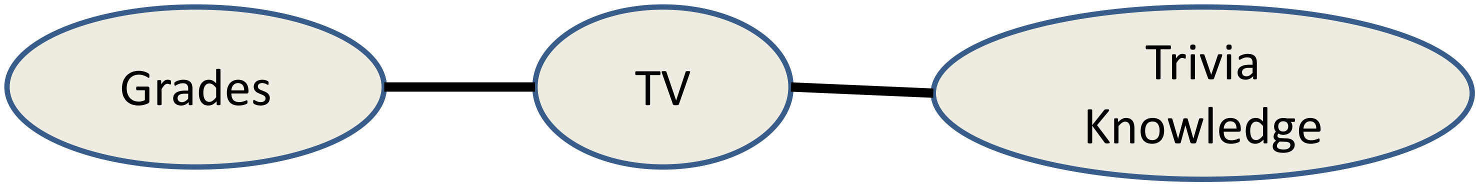
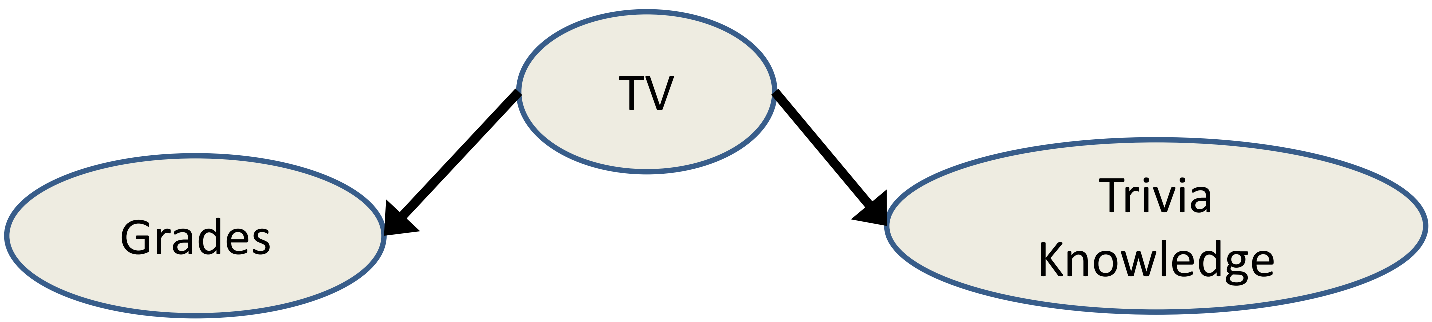
But…

- P(Grades)? Tougher.



- Need to compute Z, requires summing over Trivia Knowledge as well. Compare with Bayes Net:

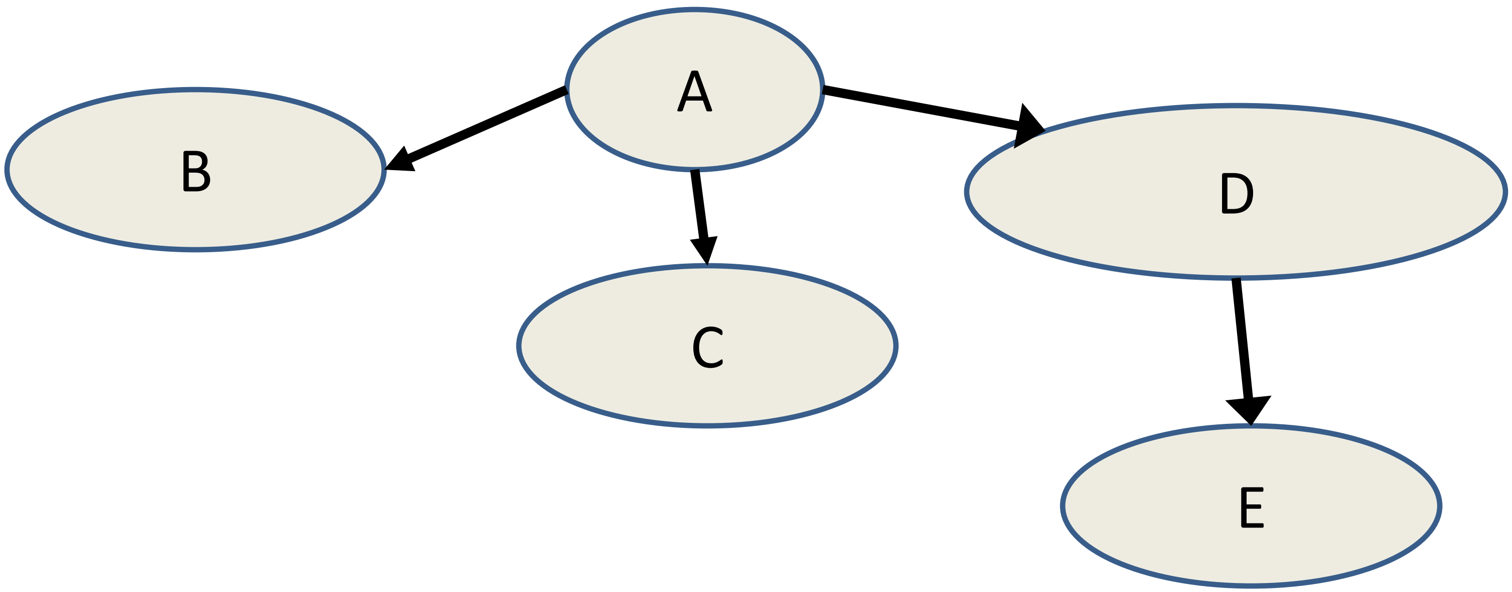


Inference in Markov Networks

- In general, we need to sum over the whole network
- A method for doing so is the *junction-tree* algorithm
  - As a side effect, it computes all the *marginals*
    - *P*(*Grades*), *P*(*TV*), *P*(*Trivia Knowledge*)
    - Key: can also compute these given evidence
  - We often want to do this for Bayes Nets too
    - Suggests a strategy: convert to Markov Network, then run junction tree algorithm

Junction Tree Algorithm

- High-level Intuition: Computing marginals is straightforward in a tree structure
- Consider a directed Bayes Net for example:

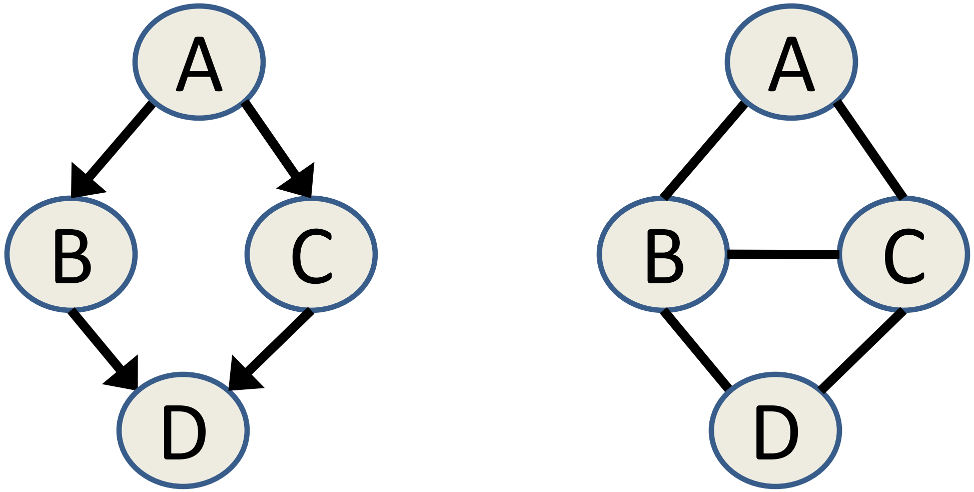


Junction Tree

- Inference of marginals is straightforward in a tree
  - Even if undirected, as we'll see
- Basic idea:
  - **If Bayes Net, convert to Markov Net**
  - Convert Markov Net into a tree structure
    - How?
      Triangulate, Build Clique Graph, Build Junction Tree
  - Do Inference on Junction Tree

Convert to Markov Net

- Consider this Bayes Net conversion:



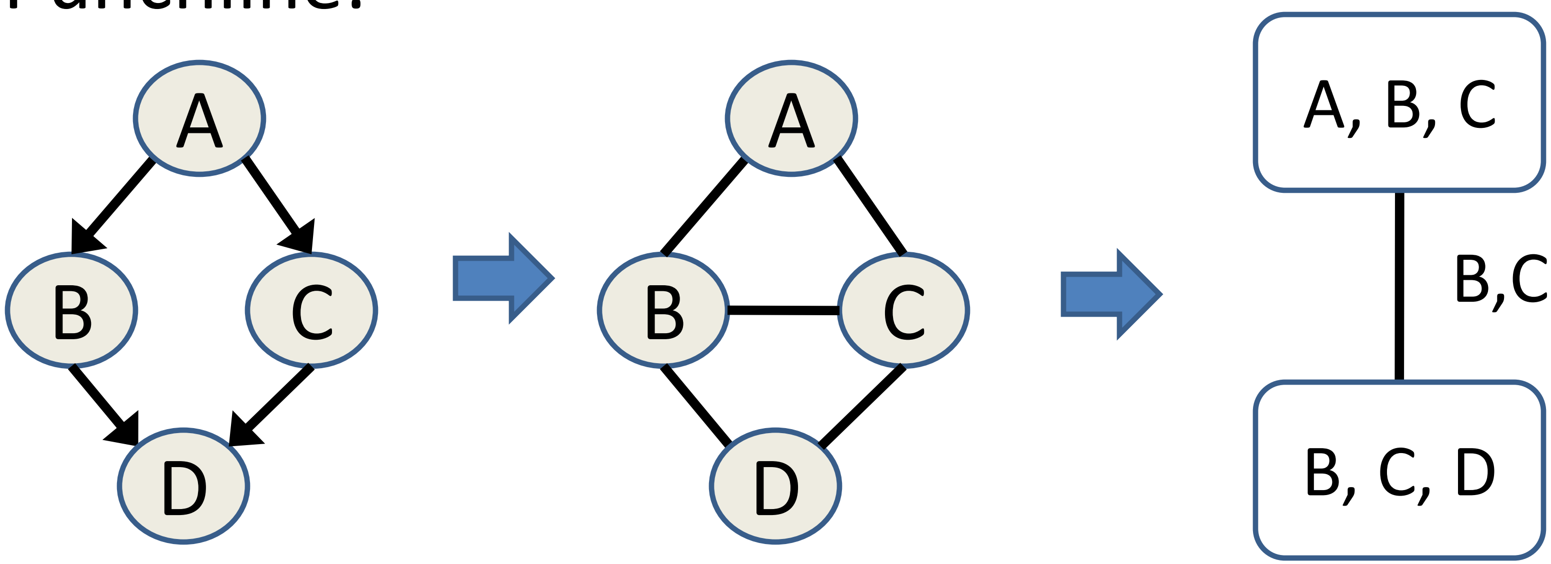
- What are the factors of the Markov Net?

Junction Tree Outline

- If Bayes Net, convert to Markov Net
- **Convert Markov Net into Junction Tree**
  - Triangulate
  - Build Clique Graph
  - Build Junction Tree
- Do Inference using Junction Tree

Convert Markov Net into Junction Tree

- Punchline:



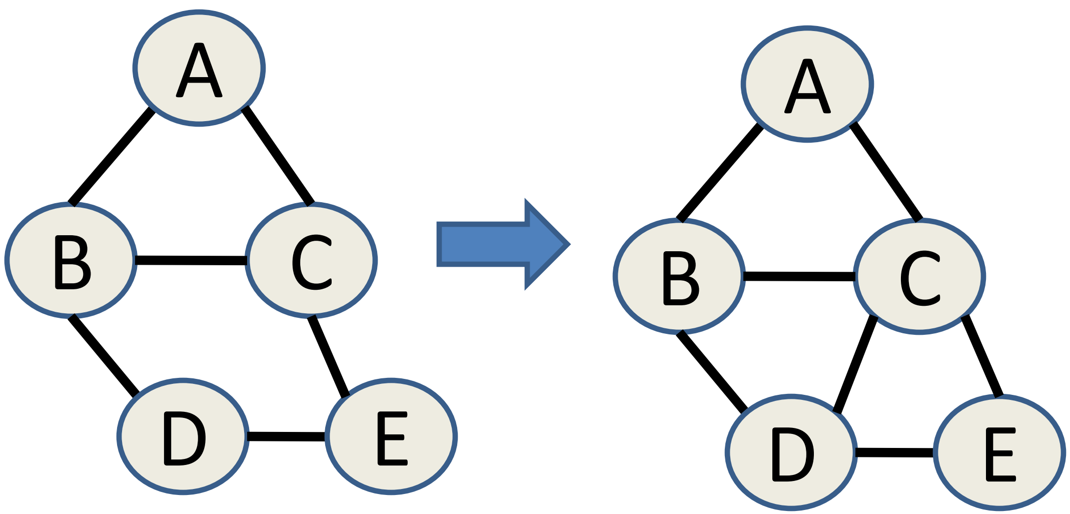
- Details follow

Junction Tree Outline

- If Bayes Net, convert to Markov Net
- Convert Markov Net into Junction Tree
  - **Triangulate**
  - Build Clique Graph
  - Build Junction Tree
- Do Inference using Junction Tree

Triangulation => “Chordal” Graph

- Goal: Every cycle of length > 3 has a chord

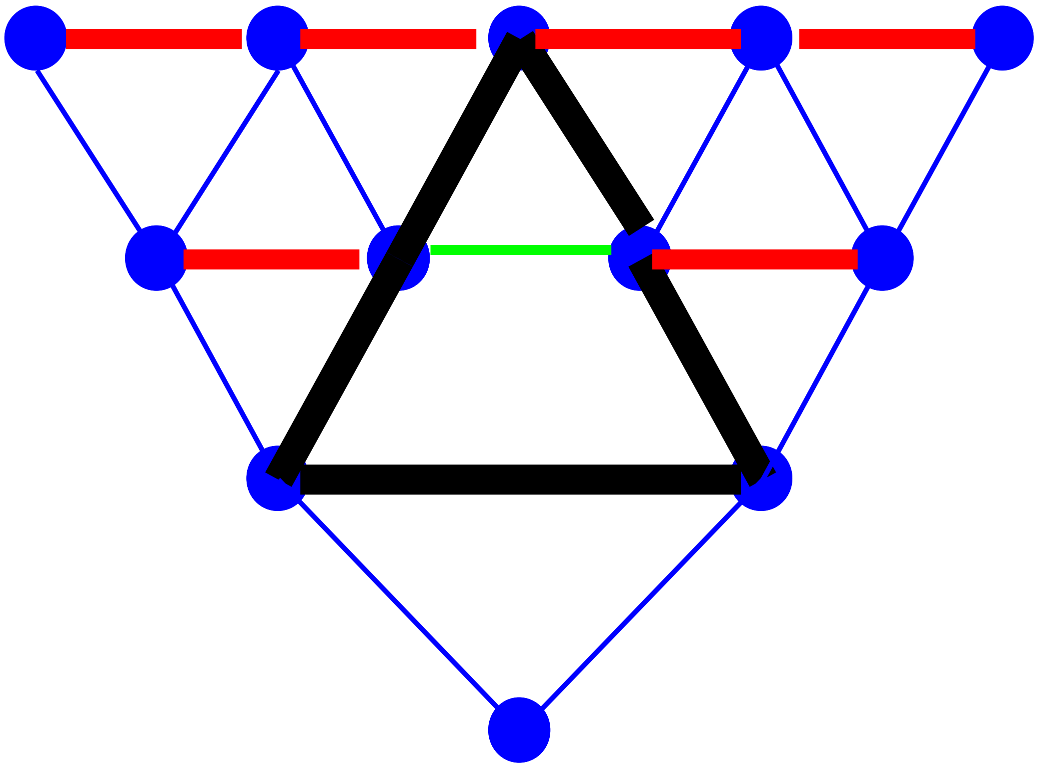


- Why? Stay tuned.

Triangulation Algorithm

Repeat while there exists a cycle of length > 3 with no chord:

Add a chord (edge between two non-adjacent vertices in such a cycle).



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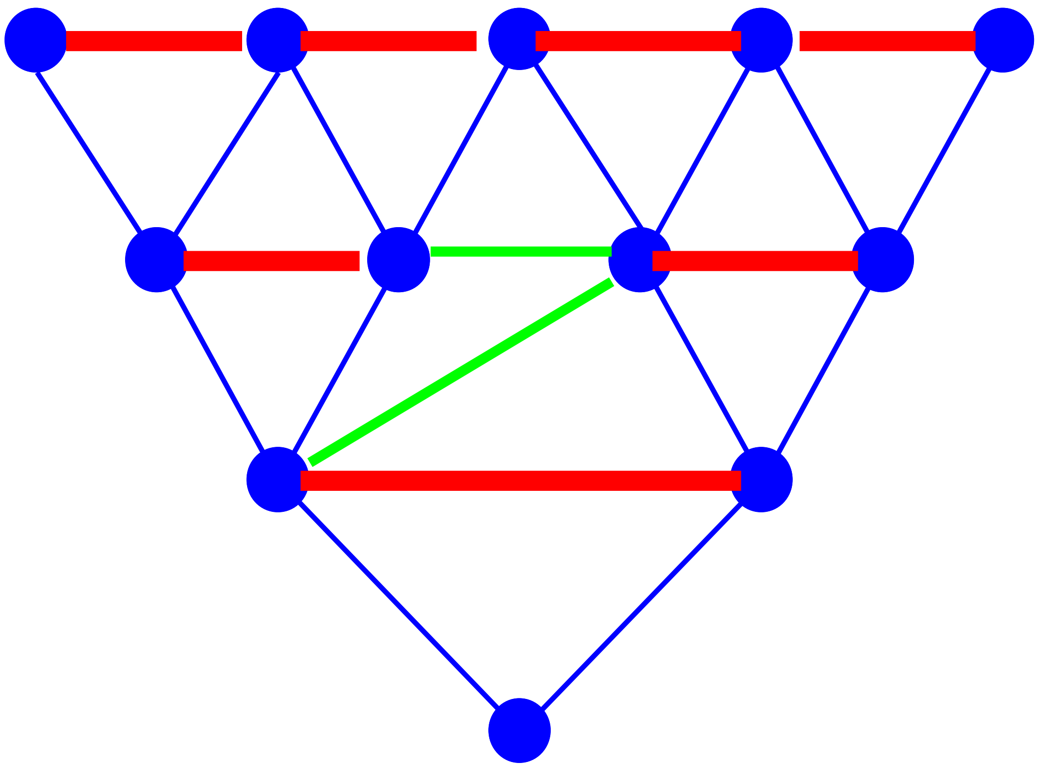
Triangulation Checking (1 of 3)

It appears to be triangulated, but how can we be sure?



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Triangulation Checking (2 of 3)

Input: Graph *G* with *n* nodes

Output: “Is *G* triangulated?”

Algorithm:

Choose any node, label it 1

for *i* = 2 to *n*

Find node with most labeled neighbors, label it *i*

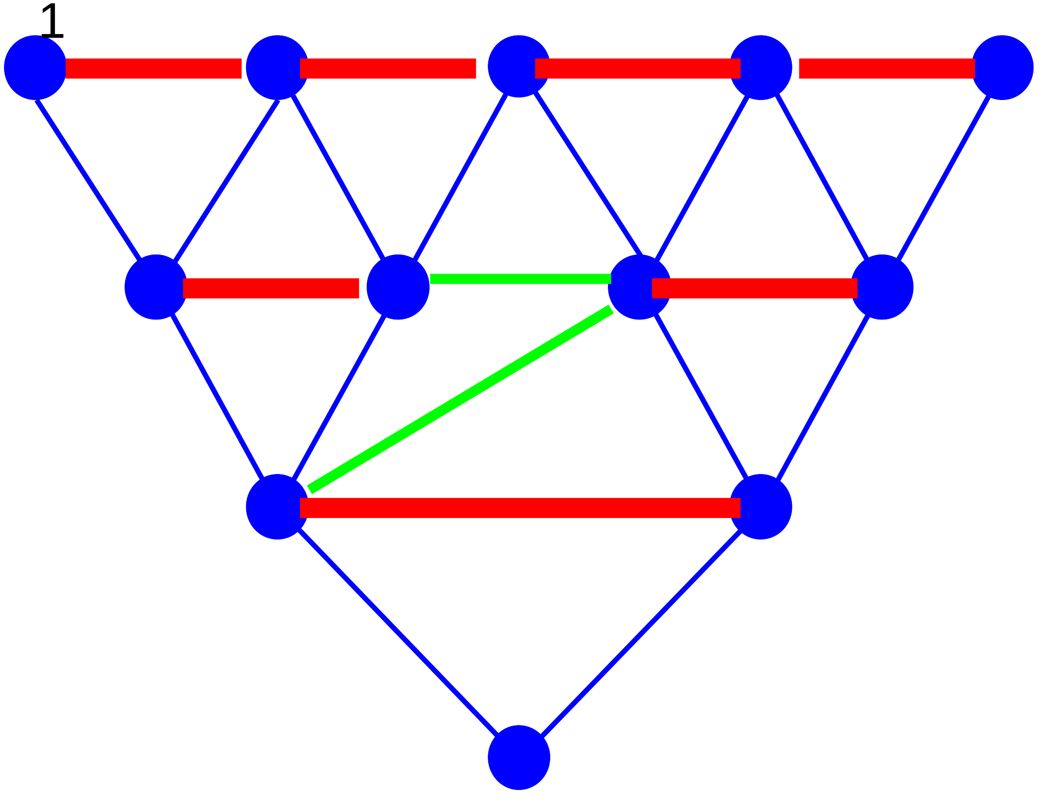
if *i* has two non-adjacent labeled neighbors

return false

return true

Triangulation Checking (3 of 3)

It appears to be triangulated, but how can we be sure?



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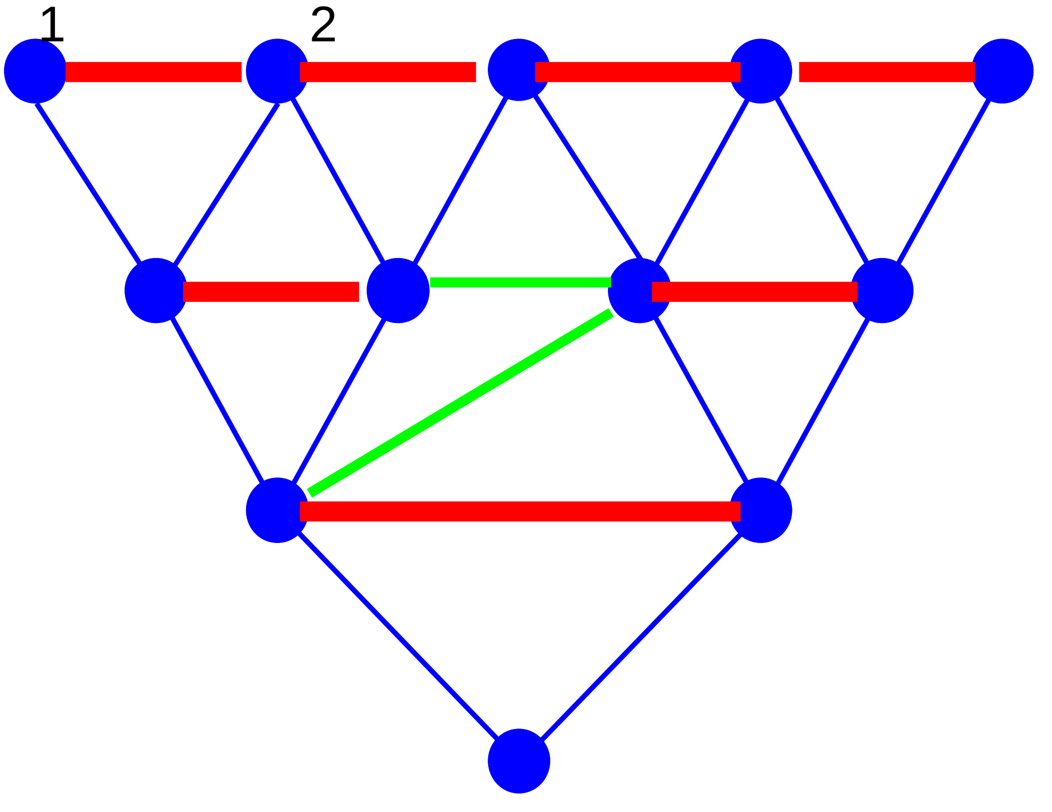

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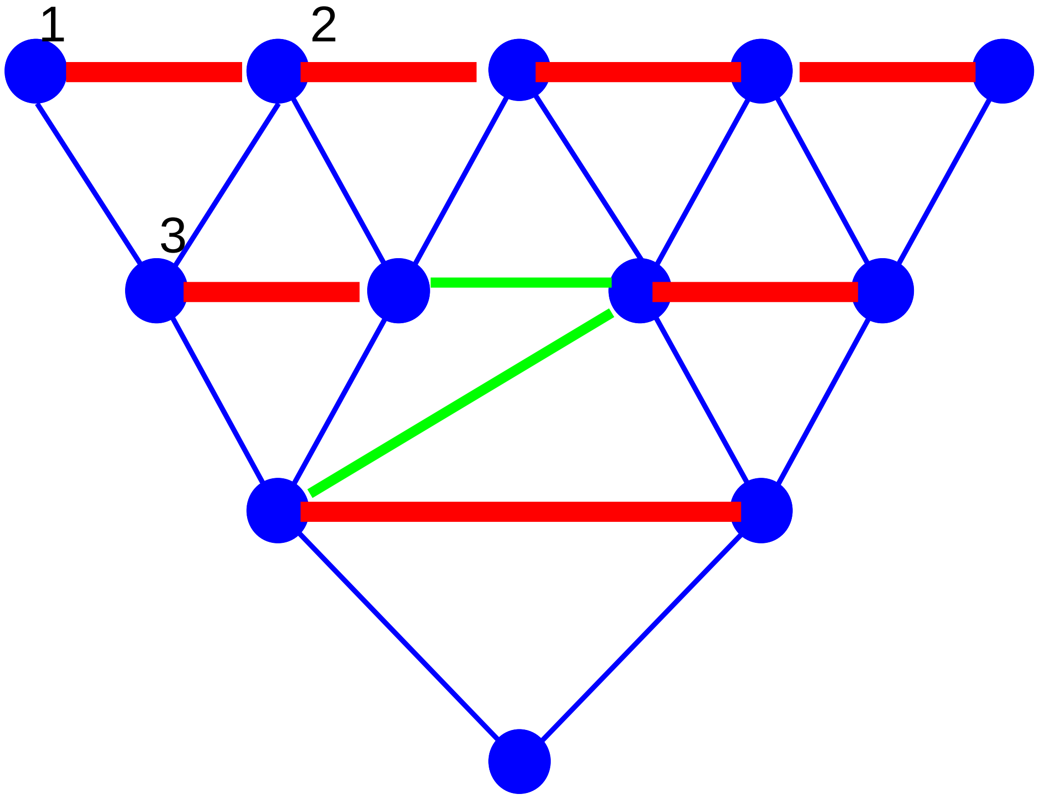

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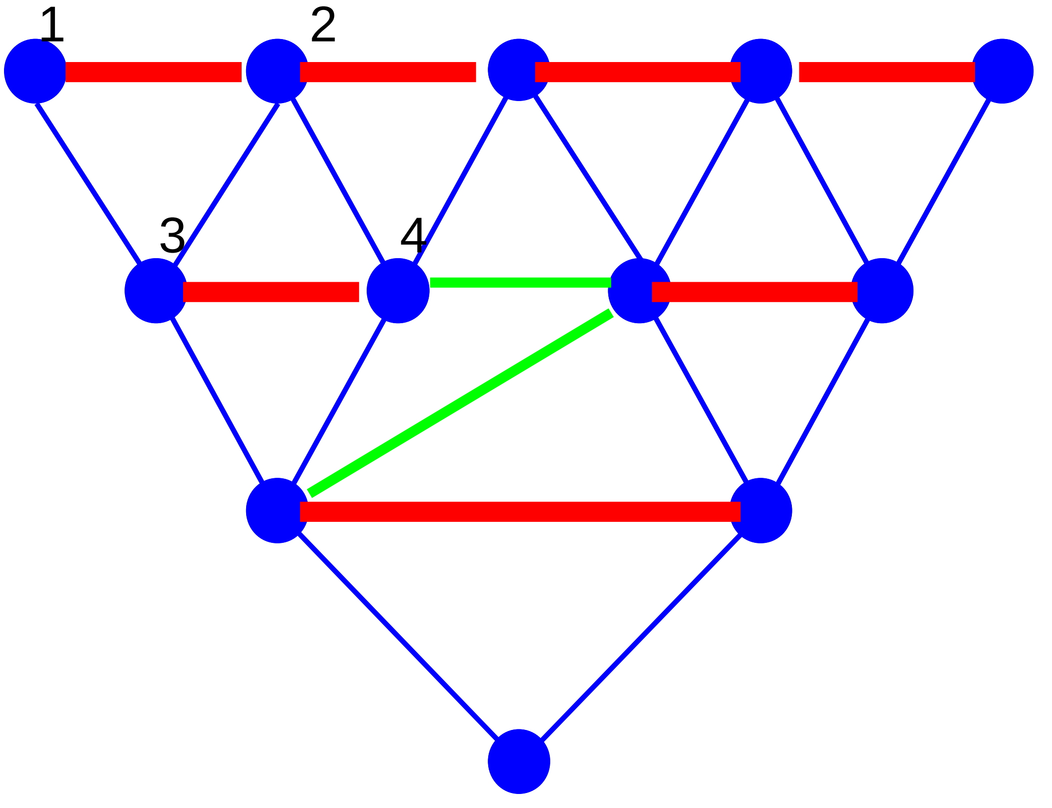

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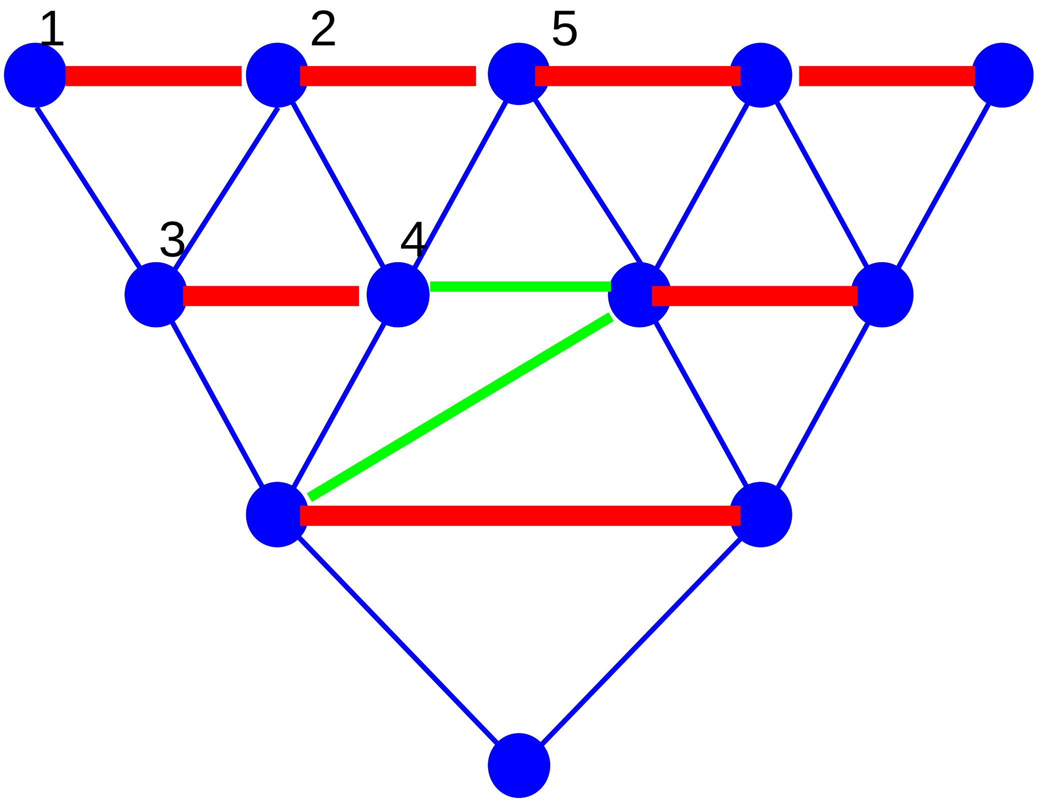

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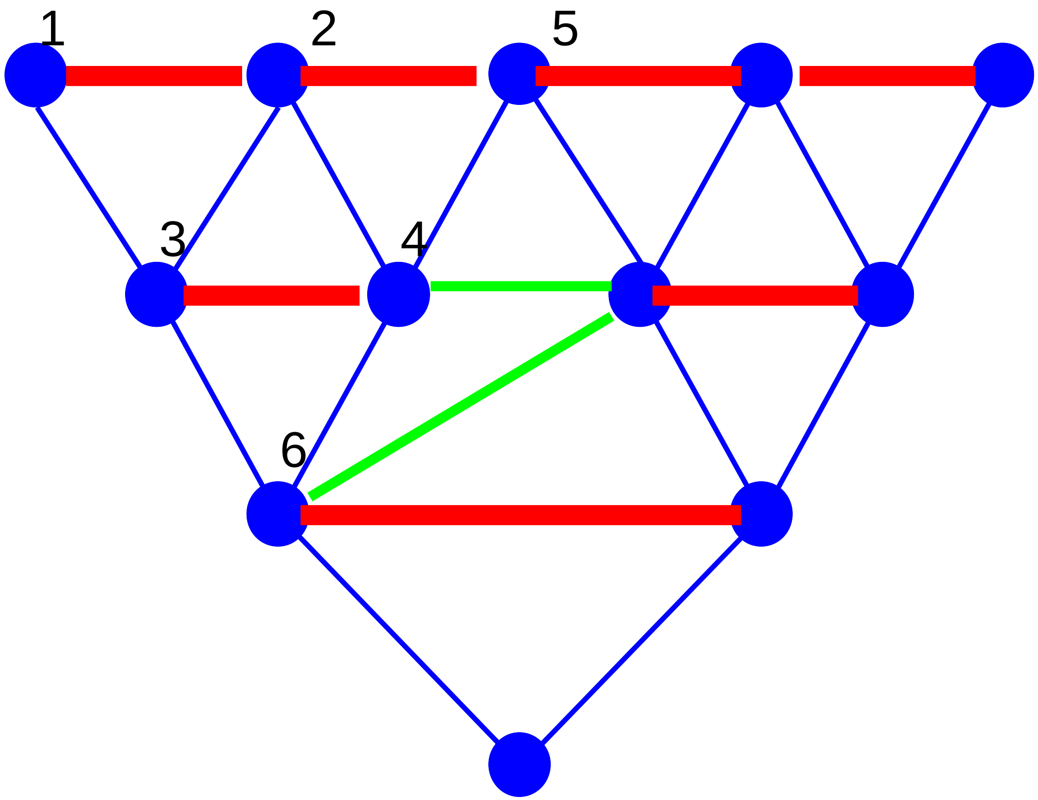

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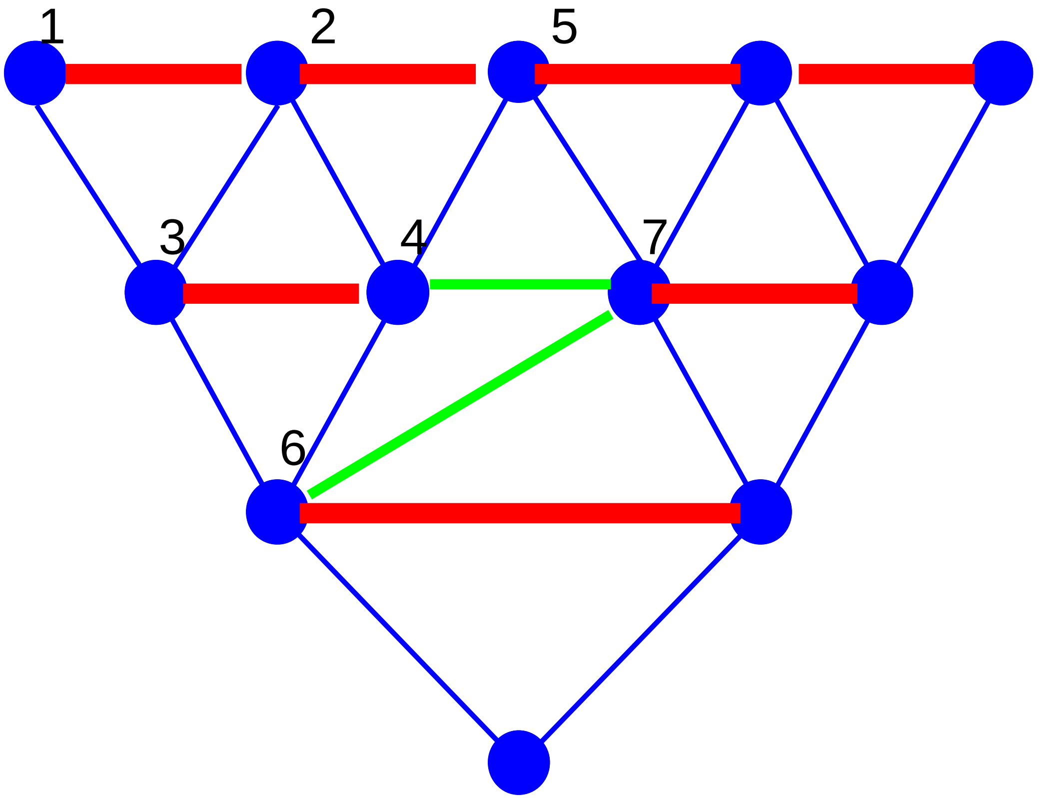
Triangulation Checking (3 of 3)

No edge between nodes 5 and 6, both of which are parents of 7.



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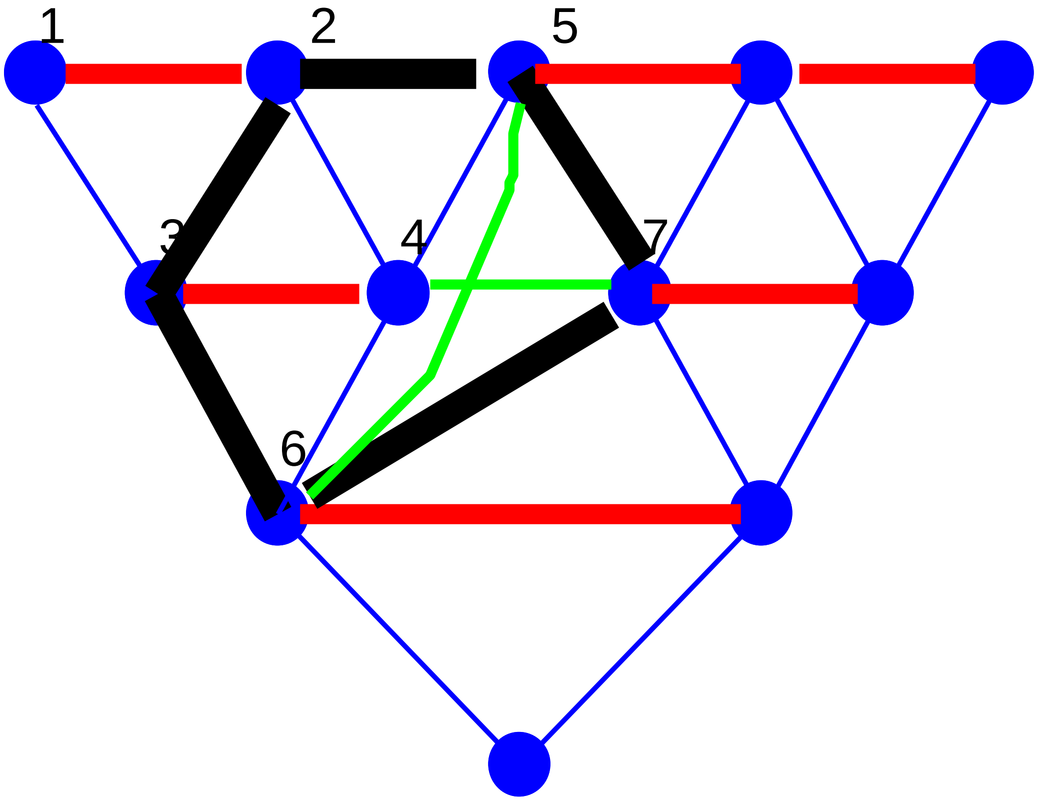
Connect the two offending nodes



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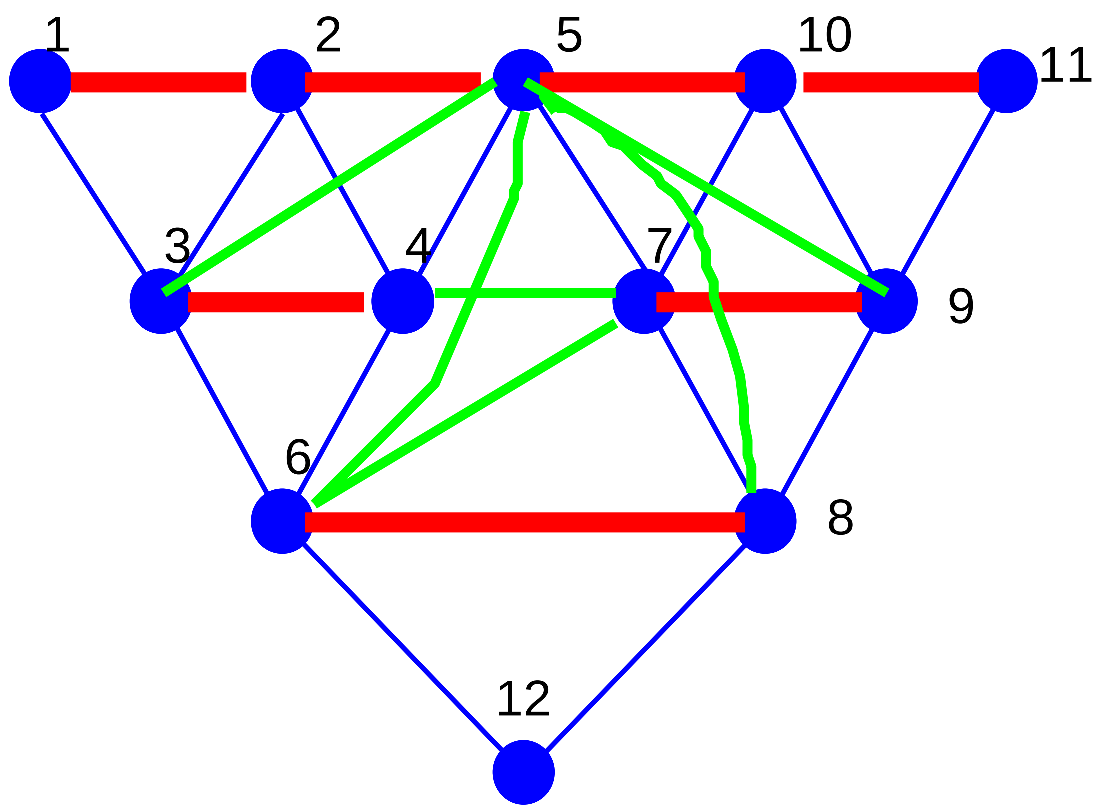
Repeat Until Triangulation Check Succeeds



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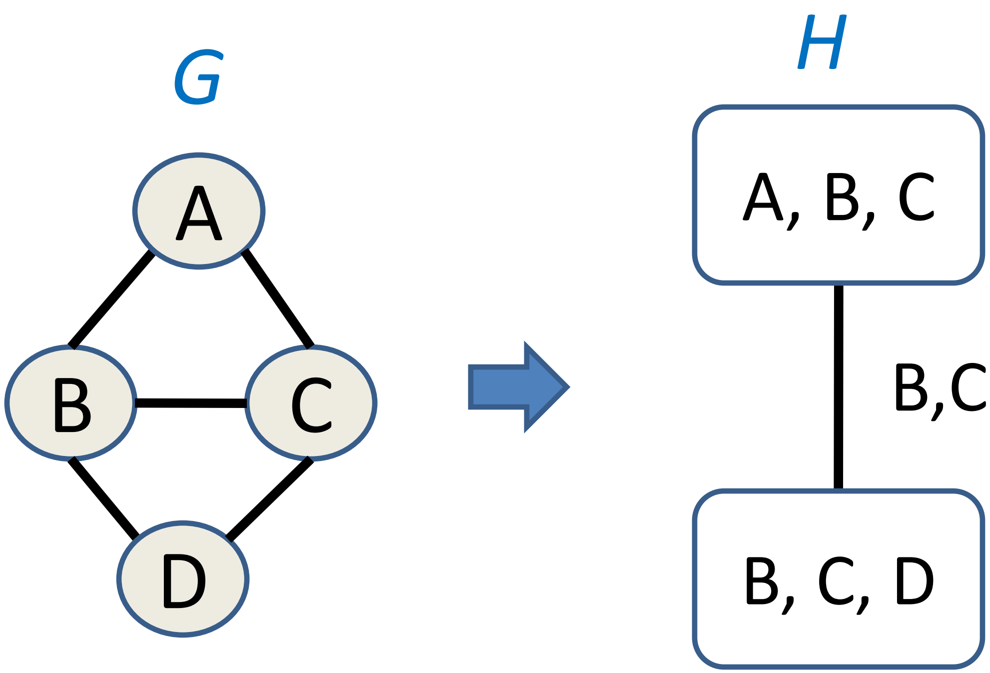
Junction Tree Outline

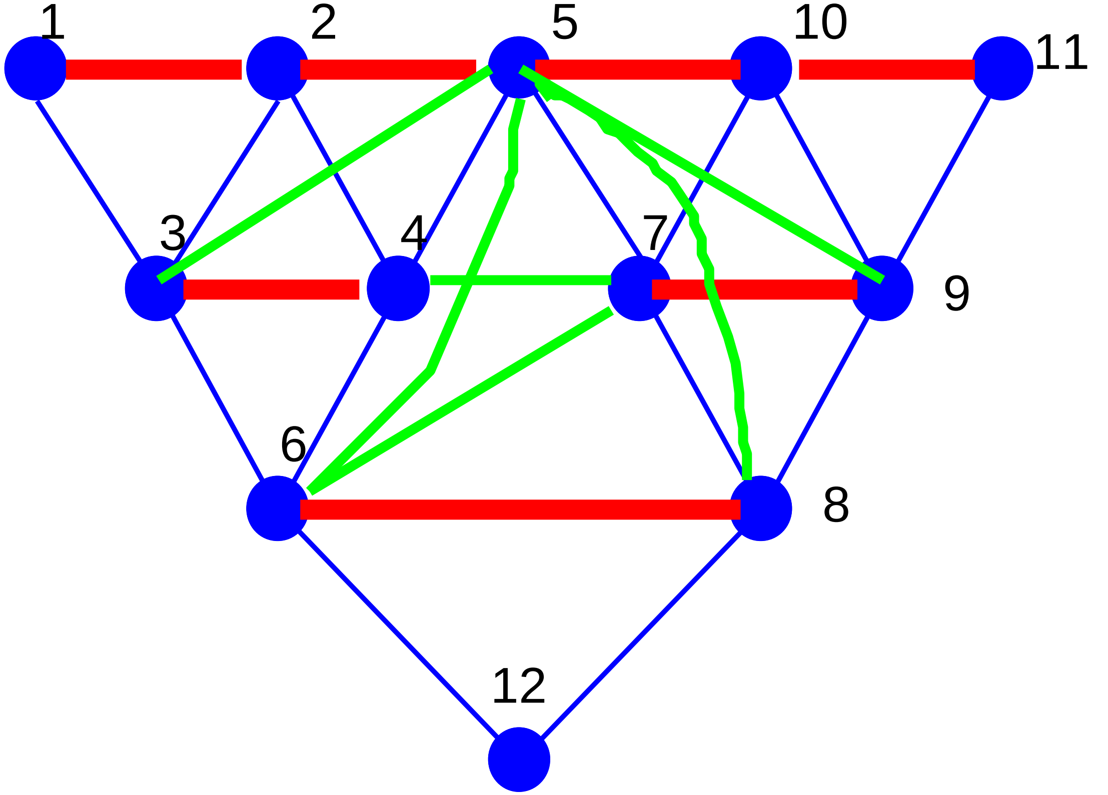
- If Bayes Net, convert to Markov Net
- Convert Markov Net into Junction Tree
  - Triangulate
  - **Build Clique Graph**
  - Build Junction Tree
- Do Inference using Junction Tree

Building Clique Graph *H*

- Create a **node** in *H* for each maximal clique in *G*
- Create **edges** in *H* between adjacent cliques in *G*
  - Convenience: Label edges in *H* with nodes' intersection



Bigger Example



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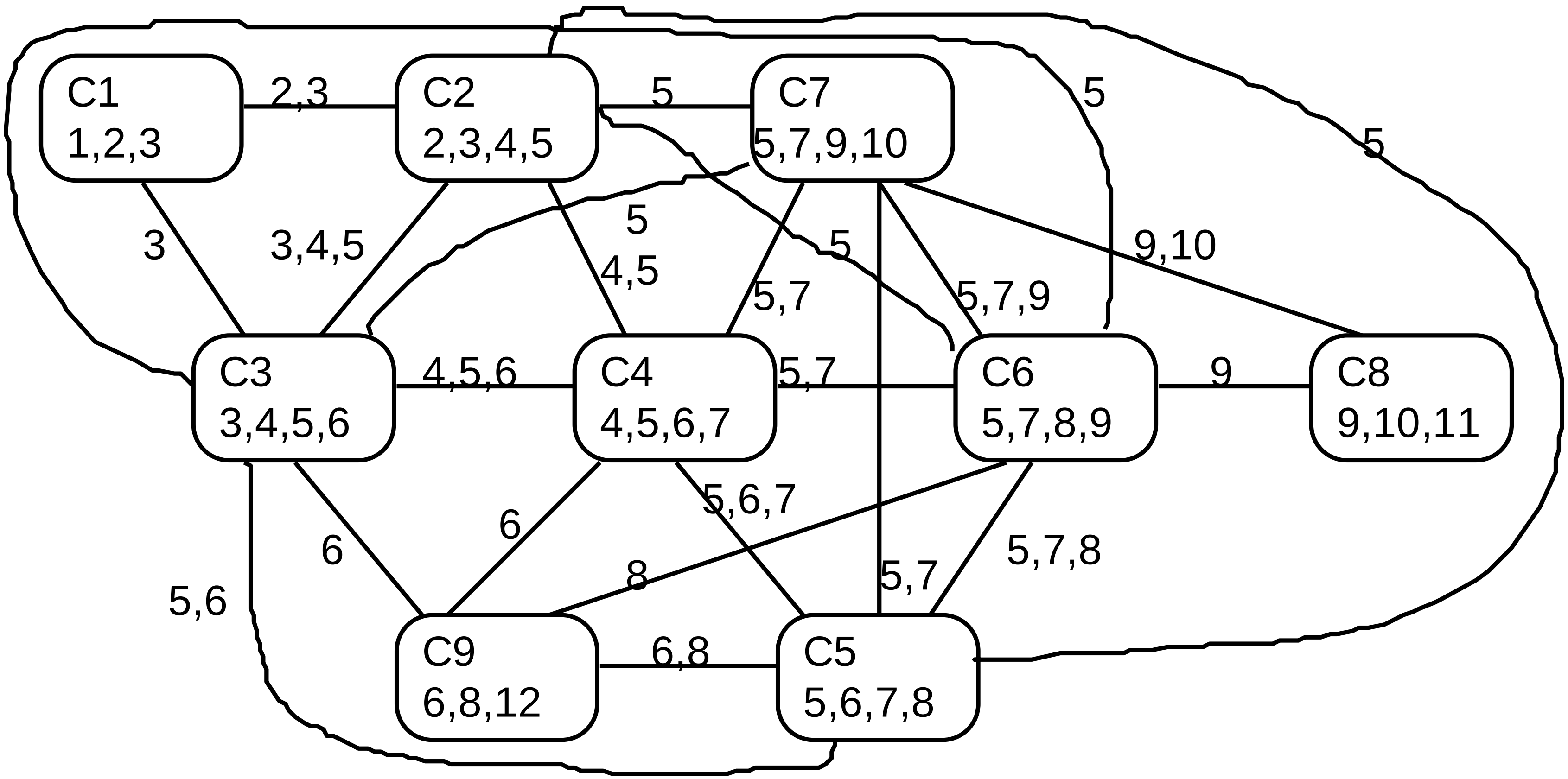
Bigger Example – Clique Graph



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Junction Tree Outline

- If Bayes Net, convert to Markov Net
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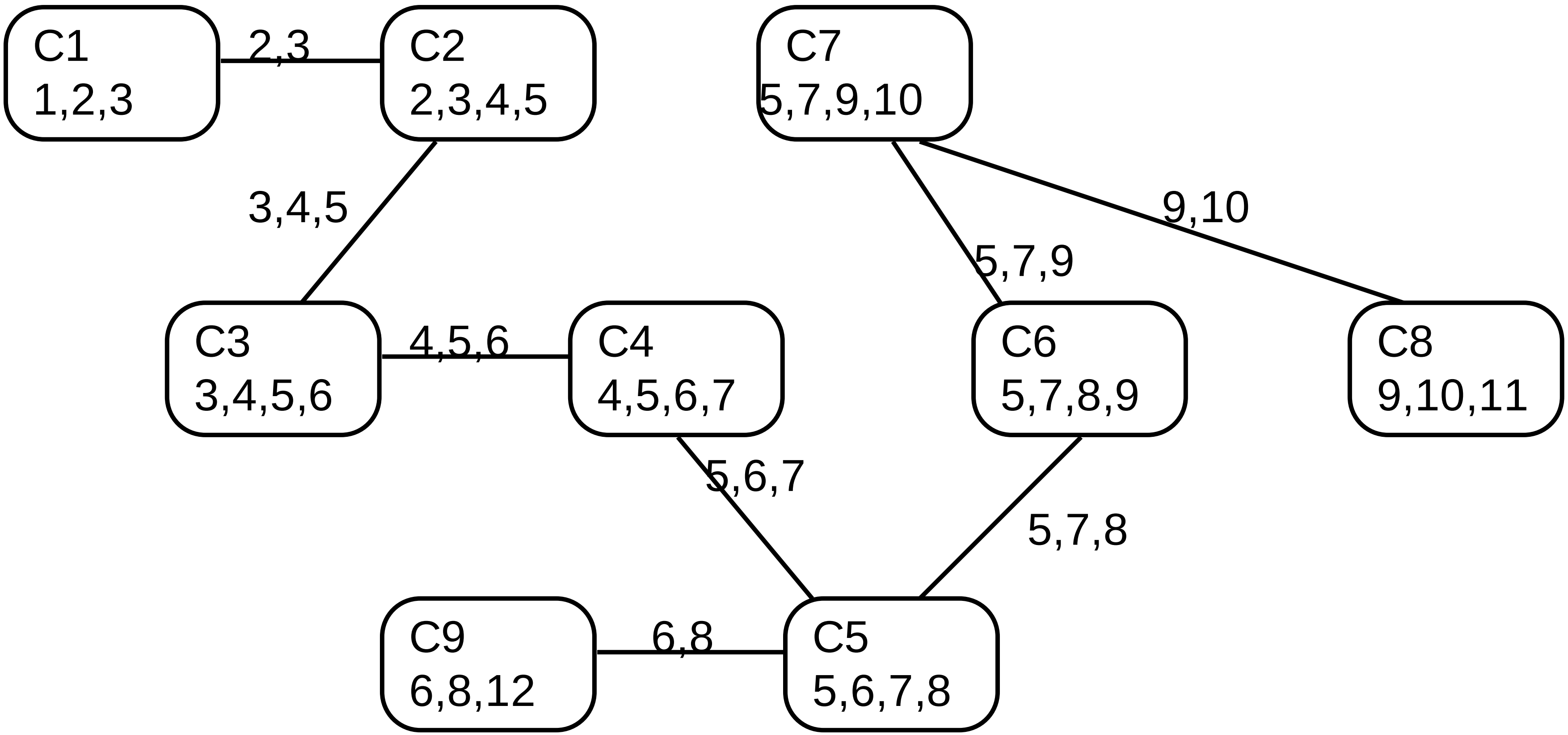
Build Junction Tree

- A **Junction Tree** is a subgraph of the clique graph that
  - Is a tree
  - Contains all the nodes of the clique graph
  - Satisfies the **junction tree property**
    - For each pair of cliques U, V with intersection S, all cliques on path between U and V contain S

Junction Tree Example



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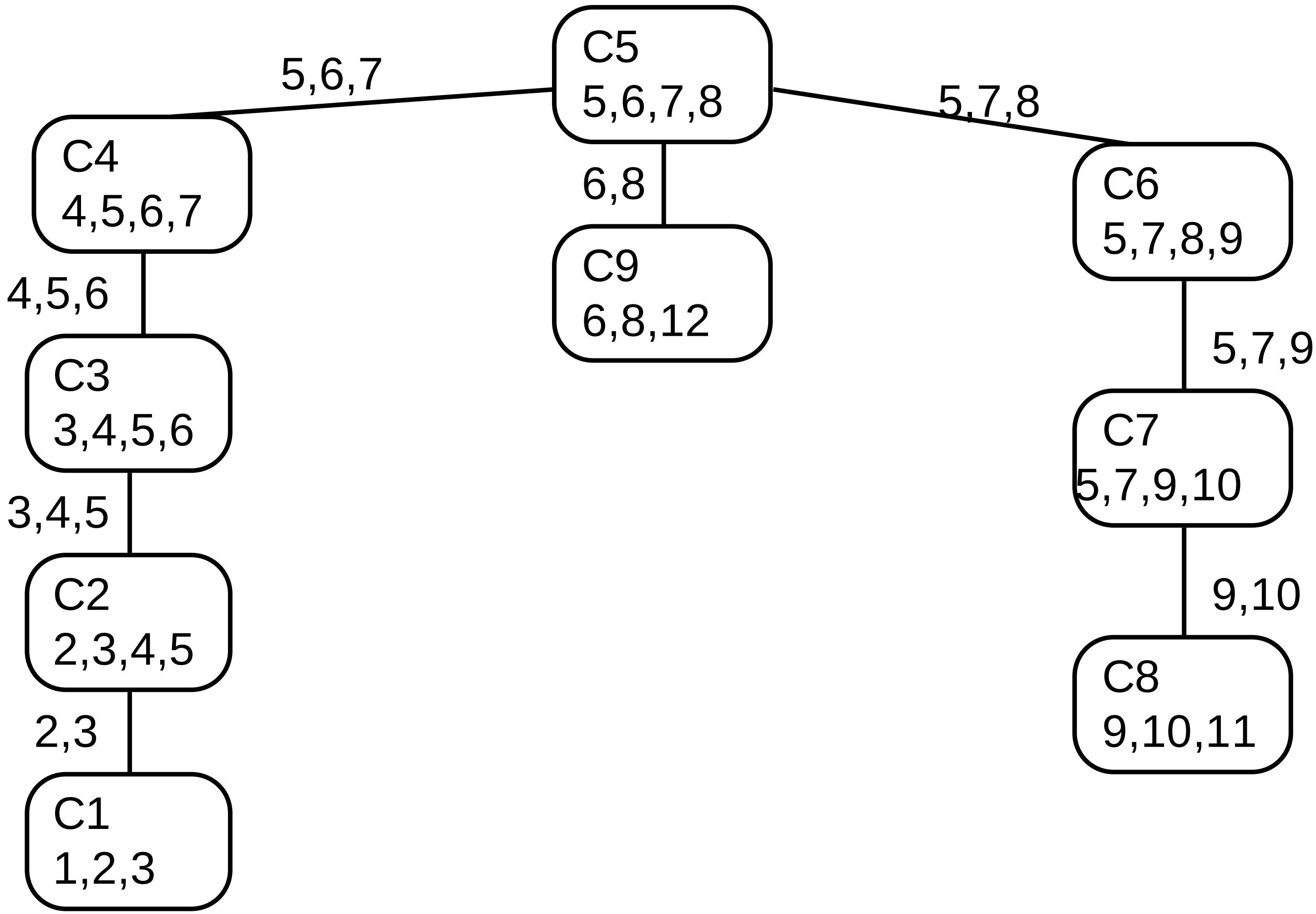
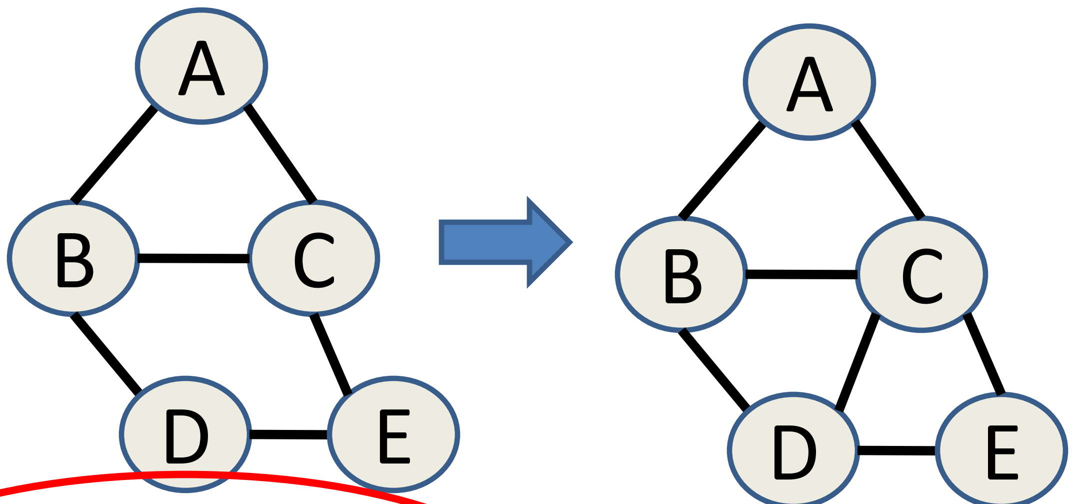
Choose a Root



Remember This?

- Goal: Every cycle of length > 3 has a chord



- Why? Stay tuned.

Can we always find a Junction Tree?

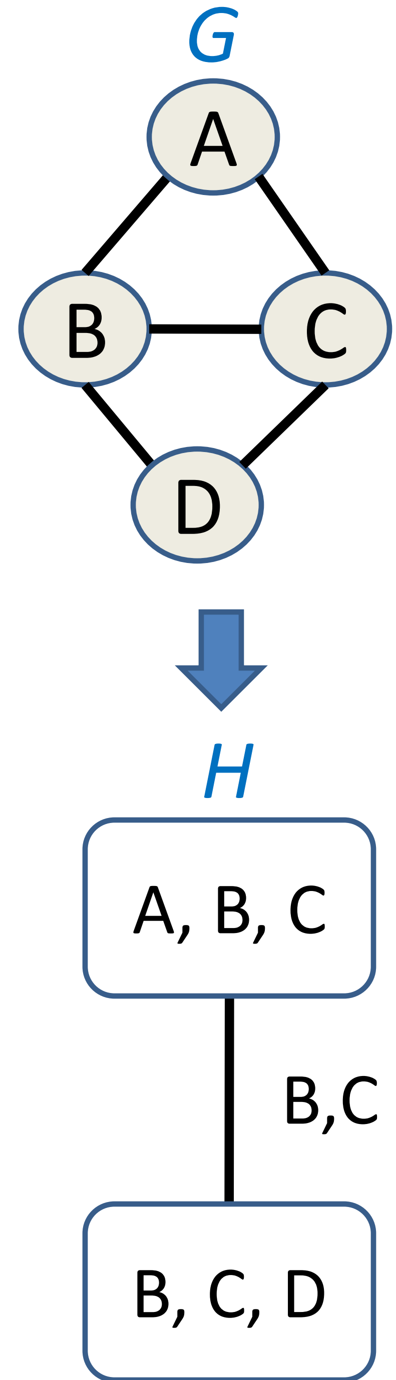
- Yes, for clique graphs of triangulated graphs
- Define "edge weight" on the clique graph to be the size of the intersection
  - Then a maximum-weight spanning tree is a junction tree
    [Jensen & Jensen, 1994]

Junction Tree

- If Bayes Net, convert to Markov Net
- Convert Markov Net into Tree
  - Triangulate
  - Build Clique Graph
  - Build Junction Tree
- **Do Inference on Tree**

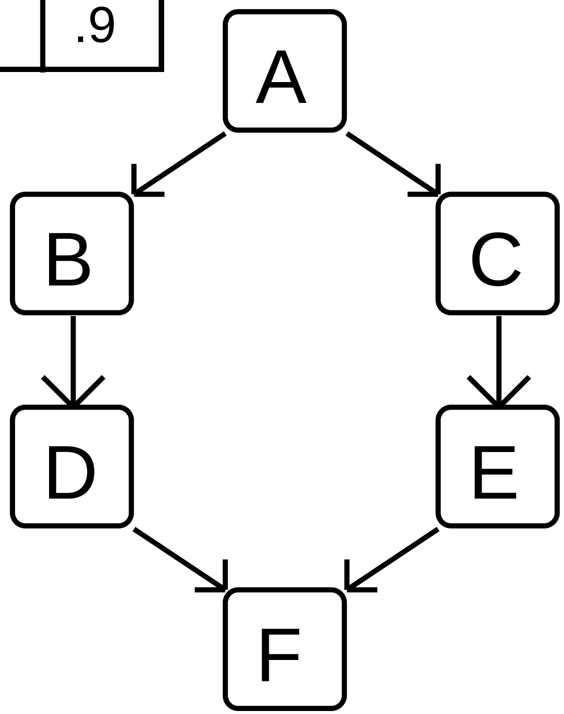
Inference

- **Initialize** clique nodes
  - Clique node in *H* is a table assigning values to its variable combinations
  - Put each potential function (or CPT) in *G* into exactly one node in *H*
  - Combine by multiplying "pointwise" (as in variable elimination)



Example

a	$\neg a$
.1	.9



	a	$\neg a$
$b \mid A$	.1	.9
$\neg b \mid A$	.9	.1

	a	$\neg a$
$c \mid A$	.7	.2
$\neg c \mid A$	.3	.8

	$d \mid B$	$\neg d \mid B$
b	.4	.6
$\neg b$	.7	.3

	$e \mid C$	$\neg e \mid C$
c	.5	.5
$\neg c$	.4	.6

	d		$\neg d$	
	e	$\neg e$	e	$\neg e$
$f \mid D, E$	.1	.5	.4	.8
$\neg f \mid D, E$	.9	.5	.6	.2

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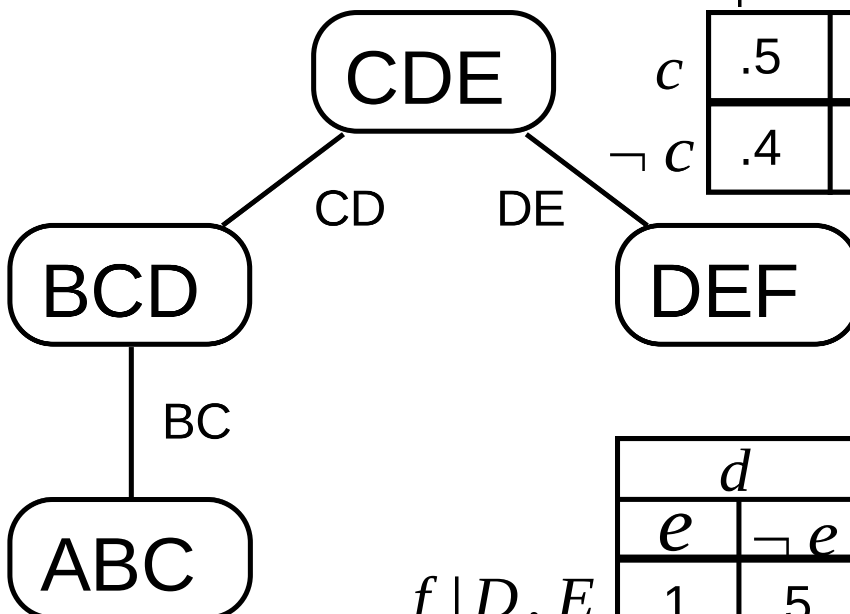
Junction Tree with CPTs



$\mathbf{P}(D|B)$

	$d\|B$	$\neg d\|B$
b	.4	.6
$\neg b$	.7	.3

$\mathbf{P}(E|C)$

	$e\|C$	$\neg e\|C$
c	.5	.5
$\neg c$	.4	.6

$\mathbf{P}(A,B,C)$

	b		$\neg b$	
	c	$\neg c$	c	$\neg c$
a	.007	.003	.063	.027
$\neg a$	.162	.648	.018	.072

$\mathbf{P}(F|D,E)$

	d		$\neg d$	
	e	$\neg e$	e	$\neg e$
$f\|D,E$	.1	.5	.4	.8
$\neg f\|D,E$	.9	.5	.6	.2

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Junction Tree Algorithm

- Incorporate Evidence -- For each $E = e$
  - Find one junction tree node containing E
  - Zero out all cells with $E \neq e$
- Upward Pass (from leaves to root)
  - Each leaf sends message to parent
    - Message = leaf's table after summing out variables not in parent
  - Parent propagates message
    - Multiplies in the child's message, then repeats process

Junction Tree Algorithm

- Downward Pass
  - Root sends child a message
    - Divides its table by child’s message from upward pass
    - Sums out variables not in child, and sends
  - Child propagates the message
    - After multiplying in parent’s message, child’s table is the joint distribution over its variables
    - Child continues the process (acts as root)

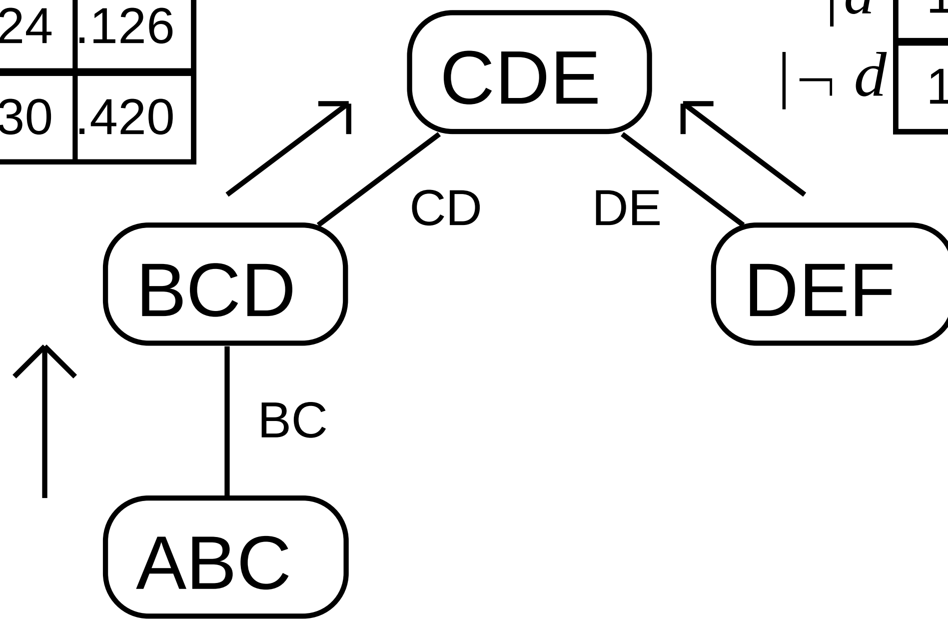
Upward Pass – assume no evidence

$\mathbf{P}(C, D)$

	d	$\neg d$
c	.124	.126
$\neg c$	.330	.420

$\mathbf{P}(B,C)$

	c	$\neg c$
b	.169	.651
$\neg b$	.081	.099



$\mathbf{P}(|\mathrm{D}, \mathrm{E})$

	$\|e$	$\|\neg e$
$\|d$	1.0	1.0
$\|\neg d$	1.0	1.0

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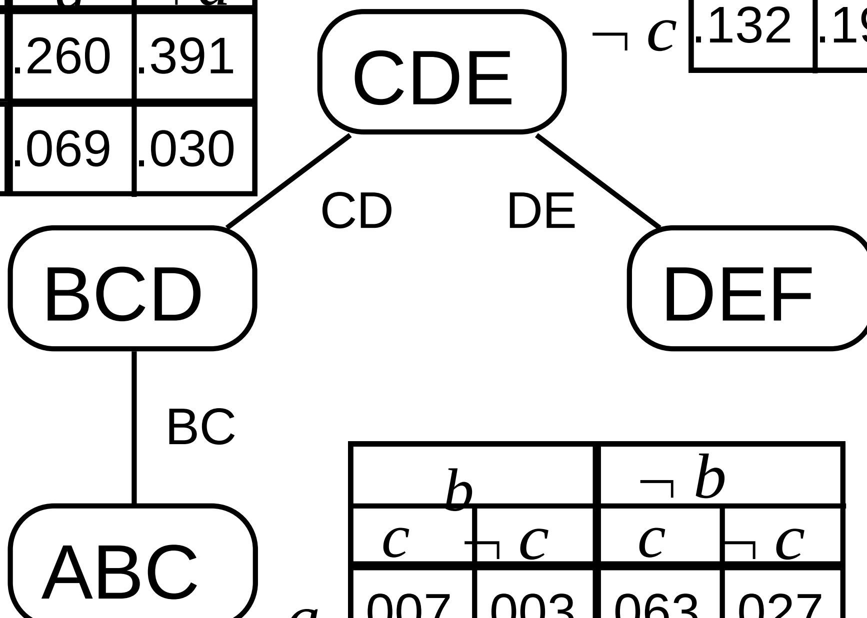
Status After Upward Pass

$\mathbf{P}(B, C, D)$

	c		$\neg c$	
	d	$\neg d$	d	$\neg d$
b	.068	.101	.260	.391
	.057	.024	.069	.030



	d		$\neg d$	
	e	$\neg e$	e	$\neg e$
c	.062	.062	.063	.063
$\neg c$	.132	.198	.168	.252

$\mathbf{P}(C, D, E)$

	d		$\neg d$	
	e	$\neg e$	e	$\neg e$
$f \mid$	.1	.5	.4	.8
$\neg f \mid$	.9	.5	.6	.2

$\mathbf{P}(F \mid D, E)$

	b		$\neg b$	
	c	$\neg c$	c	$\neg c$
a	.007	.003	.063	.027
$\neg a$	.162	.648	.018	.072

$\mathbf{P}(A, B, C)$

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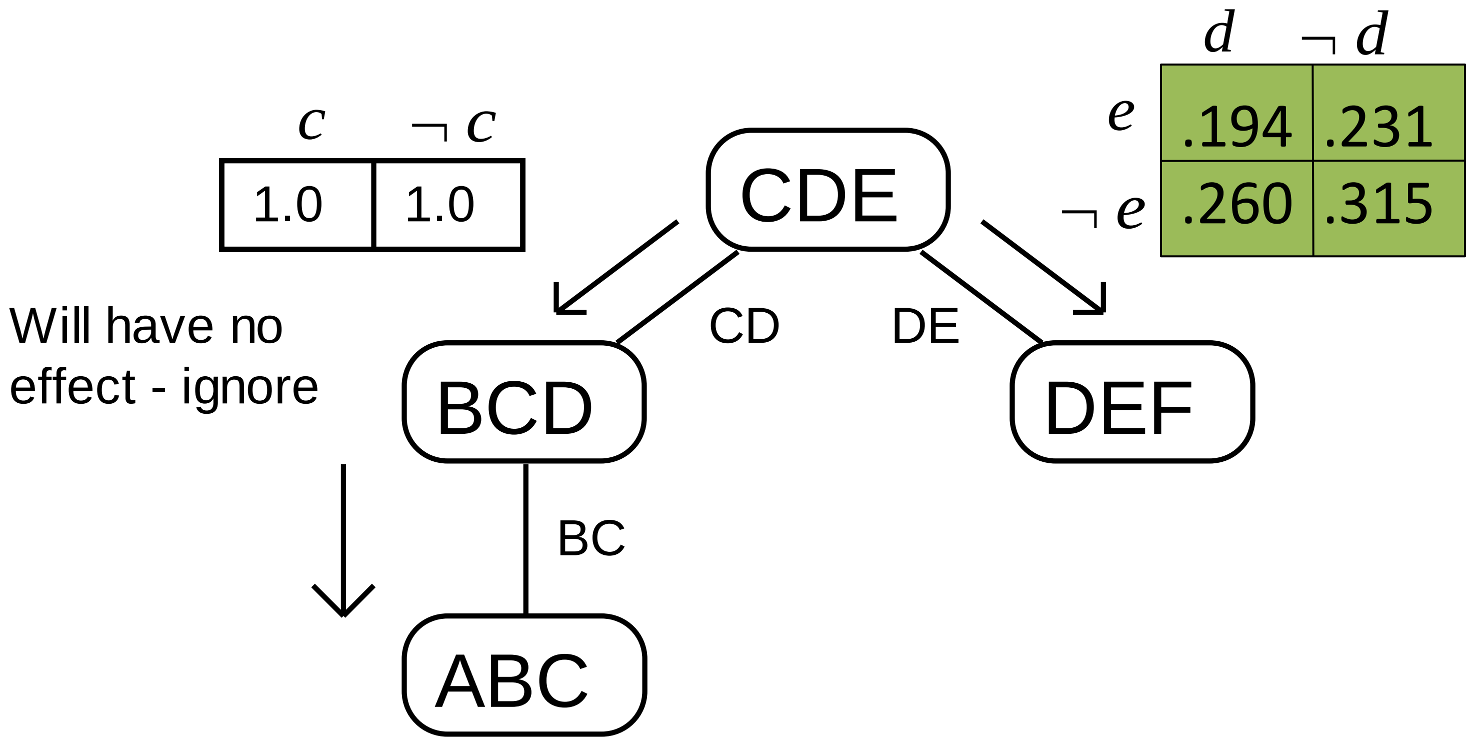
Downward Pass



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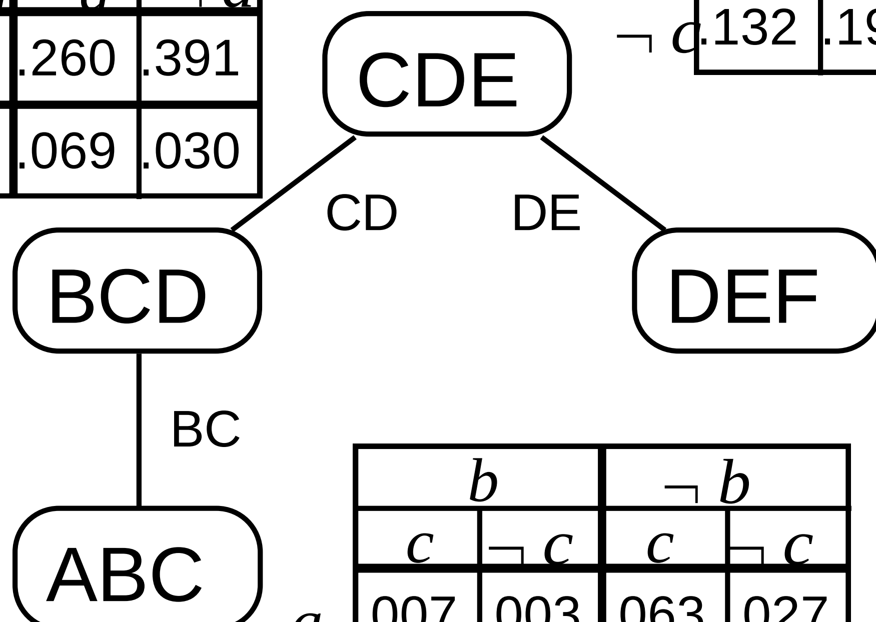
c	.062	.062	.063	.063
$\neg c$	.132	.198	.168	.252

$\mathbf{P}(C, D, E)$



	d		$\neg d$	
	e	$\neg e$	e	$\neg e$
f	.019	.130	.092	.252
$\neg f$	.175	.130	.139	.063

$\mathbf{P}(D, E, F)$

	b		$\neg b$	
	c	$\neg c$	c	$\neg c$
a	.007	.003	.063	.027
$\neg a$	.162	.648	.018	.072

$\mathbf{P}(A, B, C)$

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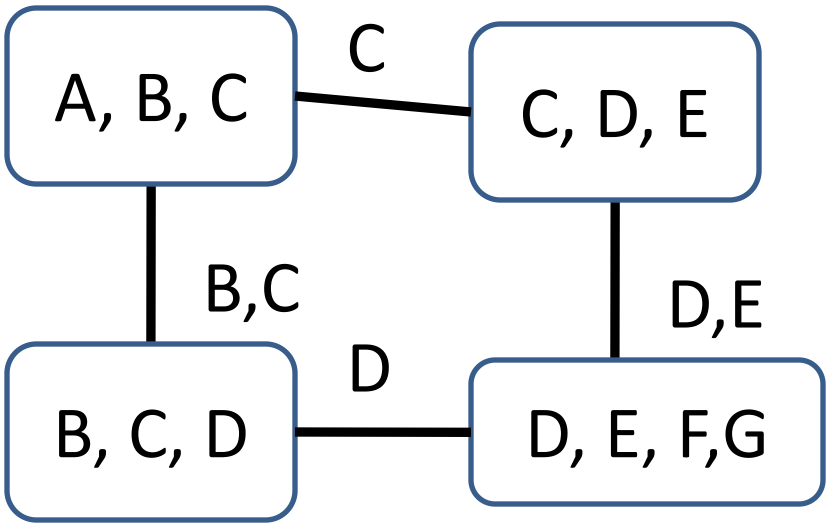
Remember Junction Tree Property

- A **Junction Tree** is a subgraph of the clique graph that
  - Is a tree
  - Contains all the nodes of the clique graph
  - Satisfies the junction tree property
    - For each pair of cliques *U*, *V* with intersection *S*, all cliques on path between *U* and *V* contain *S*

Why a Tree?

- Consider the alternative – cycles:



- Previous algorithm not applicable -- can't define upward, downward pass

Finishing touches

- We have joint distributions
  - P(A, B, C), P(C, D, E), etc.
- Compute marginals by summing out
  - Key: These sums are over small #s of variables
- If evidence changes, we repeat forward-backward pass
  - BUT we don’t have to re-compute the junction tree (= savings)