Machine Learning

Reinforcement Learning

(slides from Bryan Pardo, Ian Horswill, Bill Smart at Washington University in St. Louis)

Learning Types

- Supervised learning:
  - (Input, output) pairs of the function to be learned can be perceived or are given.

    Back-propagation in Neural Nets

- Unsupervised Learning:
  - No information about desired outcomes given

    K-means clustering

- Reinforcement learning:
  - Reward or punishment for actions

    Q-Learning

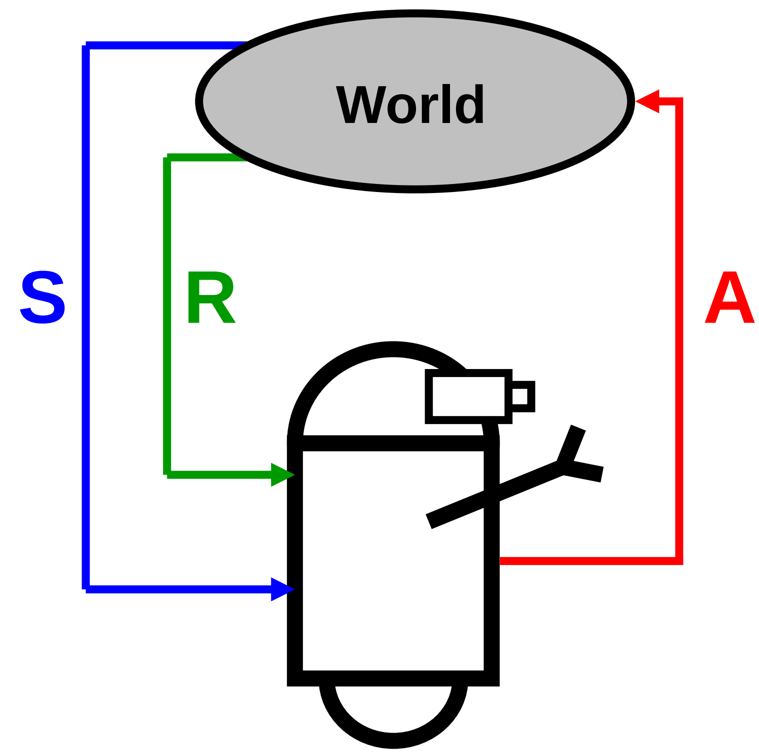
Reinforcement Learning

- Task
  - Learn how to behave to achieve a goal
  - Learn through experience from trial and error

- Examples
  - Game playing: The agent knows when it wins, but doesn’t know the appropriate action in each state along the way
  - Control: a robot can measure whether it put a dish away without breaking it, but which action(s) cause success or failure?

Basic RL Model

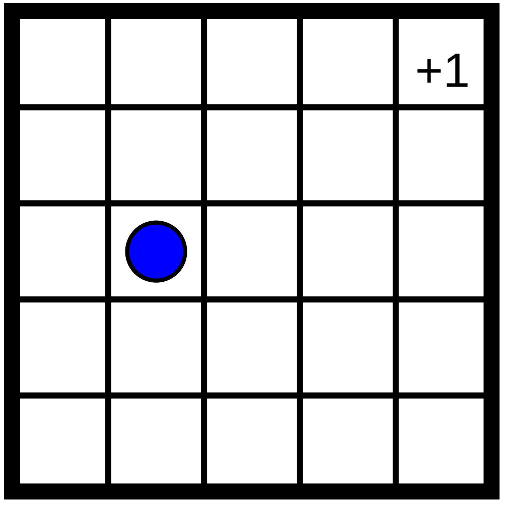
1. Observe state, s_t
2. Decide on an action, a_t
3. Perform action
4. Observe new state, s_{t+1}
5. Observe reward, r_{t+1}
6. Learn from experience
7. Repeat



•Goal: Find a control policy that will maximize the observed rewards over the lifetime of the agent

An Example: Gridworld

- Canonical RL domain
  - States are grid cells
  - 4 actions: N, S, E, W
  - Reward for entering top right cell
  - -0.01 for every other move

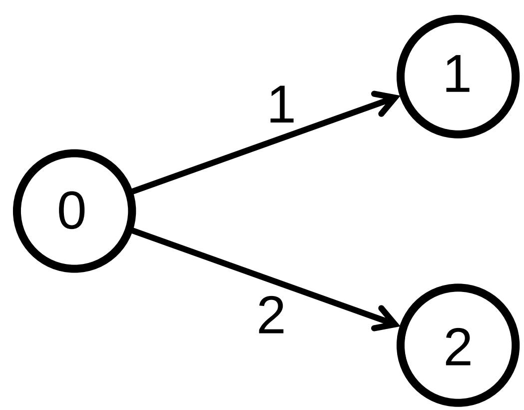


Mathematics of RL

- Before we talk about RL, we need to cover some background material
  - Simple decision theory
  - Markov Decision Processes
  - Value functions
  - Dynamic programming

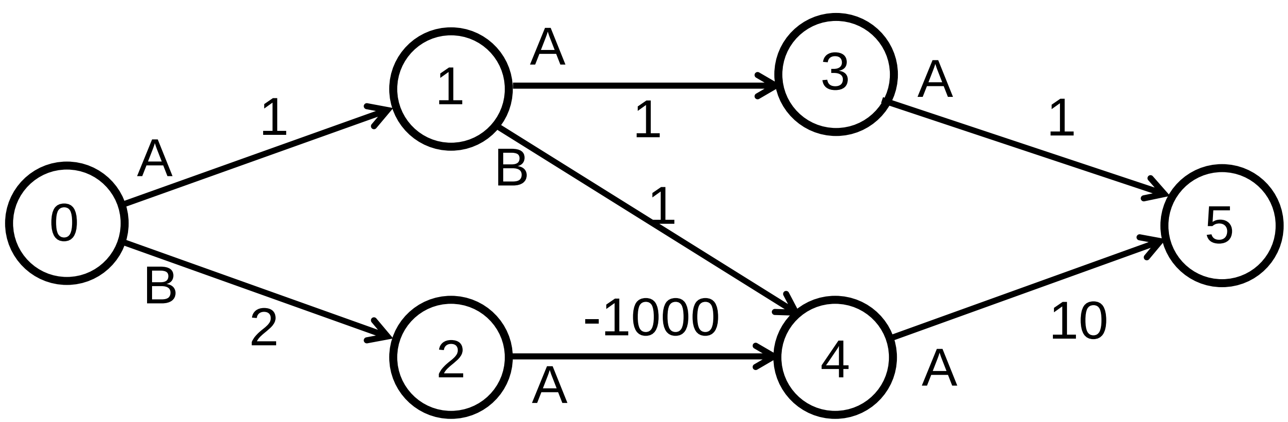
Making Single Decisions

- Single decision to be made
  - Multiple discrete actions
  - Each action has a reward associated with it
- Goal is to maximize reward
  - Not hard: just pick the action with the largest reward
- State 0 has a value of 2
  - Sum of rewards from taking the best action from the state



Markov Decision Processes

- We can generalize the previous example to multiple sequential decisions
  - Each decision affects subsequent decisions
- This is formally modeled by a Markov Decision Process (MDP)



Markov Decision Processes

- Formally, a MDP is
  - A set of states, $S = \{s_1, s_2, \ldots, s_n\}$
  - A set of actions, $A = \{a_1, a_2, \ldots, a_m\}$
  - A reward function, $R: S \times A \times S \rightarrow \Re$
  - A transition function, $P_{ij}^{a} = P(s_{t+1} = j \mid s_t = i, a_t = a)$
    - Sometimes $T: S \times A \rightarrow S$

- We want to learn a policy, $\pi: S \rightarrow A$
  - Maximize sum of rewards we see over our lifetime

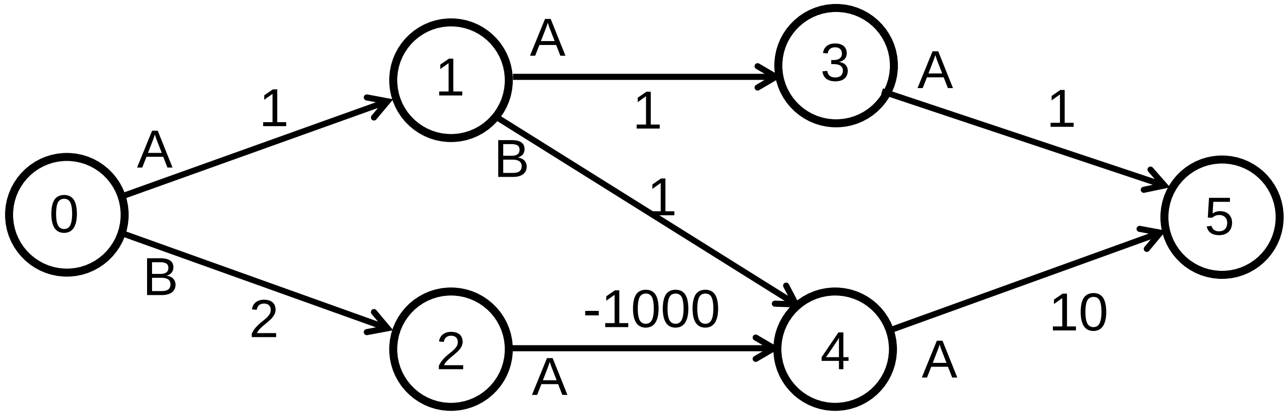
Policies

- A policy $\pi(s)$ returns what action to take in state s.
- There are 3 policies for this MDP

Policy 1: 0 →1 →3 →5

Policy 2: 0 →1 →4 →5

Policy 3: 0 →2 →4 →5



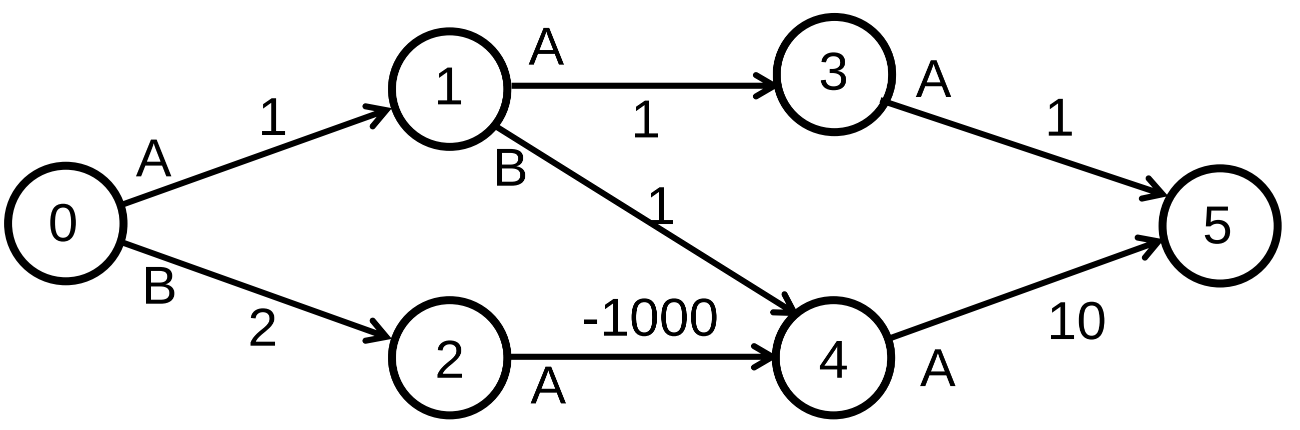
Comparing Policies

- Which policy is best?
- Order them by how much reward they see

Policy 1: 0 →1 →3 →5 = 1 + 1 + 1 = 3

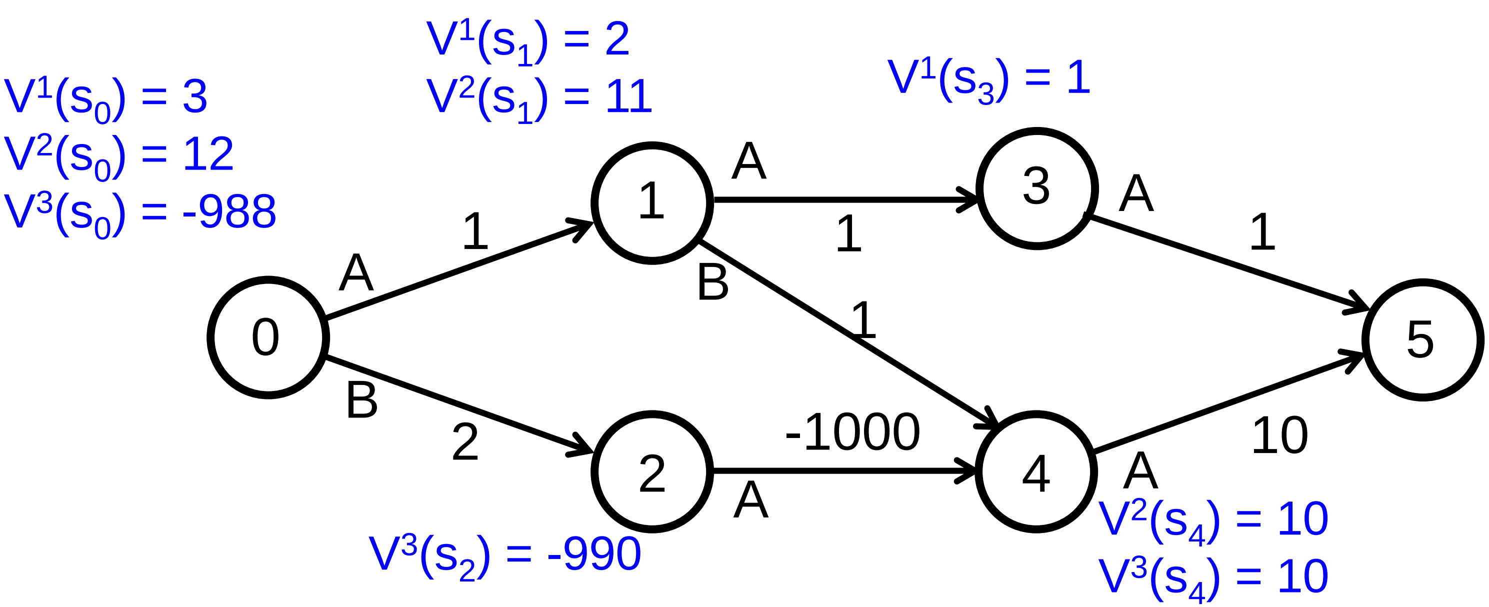
Policy 2: 0 →1 →4 →5 = 1 + 1 + 10 = 12

Policy 3: 0 →2 →4 →5 = 2 – 1000 + 10 = -988



Value Functions

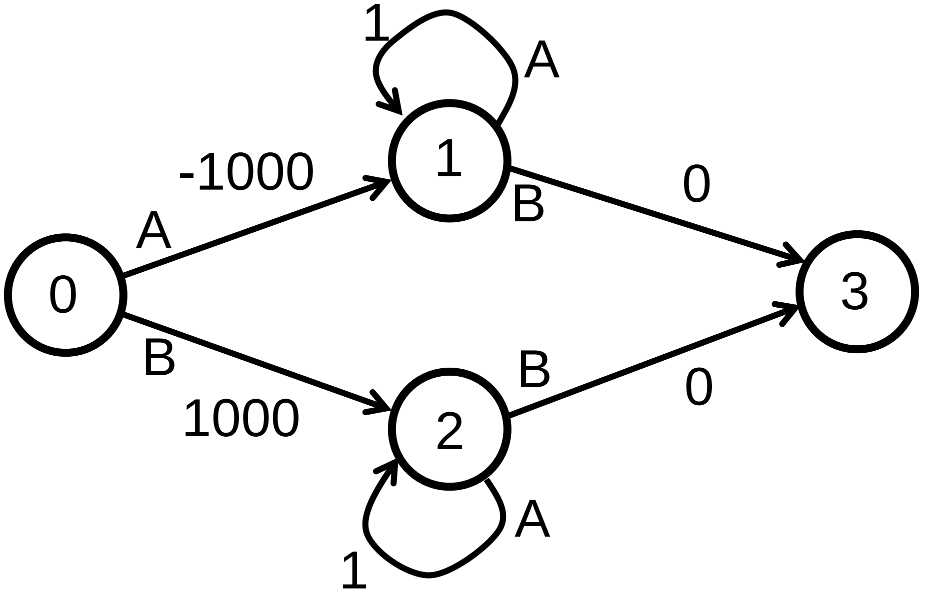
- We can associate a value with each state
  - For a fixed policy
  - How good is it to run policy π from that state s
  - This is the state value function, V



Problems with Our Function

- Consider this MDP
  - Number of steps is now unlimited because of loops
  - Value of states 1 and 2 is infinite for some policies

$$V^1(s_0) = 1 + V^1(s_0)$$



- This is bad
  - All policies with a non-zero reward cycle have infinite value

Adding up the rewards

- We had said:
  - The reward for a policy (called the **return**) is just the **sum of the rewards** you get at every time step:

$$R = r_1 + r_2 + r_3 + \cdots$$

  - And then look for a policy that **maximizes the expected** value of this sum

- But we don’t do that
  - Infinities

Discount factor

- The pure sum is **infinite** and in any case **model errors** (e.g. the agent dying) usually mean our estimates of rewards get less accurate the farther we look in the future

- So we weight future returns less by a factor γ (the **discount rate**):
$$R = r_1 + \gamma r_2 + \gamma^2 r_3 + \cdots$$

- And then our goal is to find a policy that **maximizes expected time-discounted reward**

What to do??

- A (now randomized) **policy**

$$\pi: \mathcal{S} \times \mathcal{A} \to [0,1]$$

  gives the probability of π running a given action in a given state
  - A **deterministic policy** $\pi: \mathcal{S} \to \mathcal{A}$ is a policy that in state s runs action $\pi(s)$ always
- We want to **pick a policy** that will **maximize expected reward**
- First: how do we even compute the expected reward for a *given* policy?

Value functions

- One you decide on a given policy, π, you can compute the expected return for the policy
- We express that in terms of the **state value function** for the policy
  - $V^{\pi}(s)$ is the **expected return** when starting from state s and running the policy π

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}_{ss'}^{a} \left[\mathcal{R}_{ss'}^{a} + \gamma V^{\pi}(s') \right]$$

- This **averages** over
  - All possible actions π could take from state s
  - All possible successor states s' those actions could land us in
  - All possible rewards we could get from it
- And sums for each of them
  - The reward $\mathcal{R}_{ss'}^{a}$ you get with
  - The expected return $V^{\pi}(s')$ for running the policy from the resulting state s', subject to the discount rate γ

Computing V^{π} (aka policy evaluation)

- The **naïve** thing to do is just to evaluate the definition directly:

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}^{a}_{ss'} \left[\mathcal{R}^{a}_{ss'} + \gamma V^{\pi}(s') \right]$$

- That is, we interpret it as a function definition

- Of course, this code won't work
- Why?

```
V(π, s) {
   sum = 0;
   foreach a {
      foreach s' {
          r = R[a,s,s']+ γ*V(π, s');
         sum += π(s, a)*P[a,s,s']*r;
      }
   }
   return sum;
}
```

Computing V^{π} (aka policy evaluation)

- It's an infinite recursion

```
V(π, s) {
  sum = 0;
  foreach a {
    foreach s' {
       r = R[a,s,s']+ γ*V(π, s');
       sum += π(s, a)*P[a,s,s']*r;
    }
  }
  return sum;
}
```

Computing V^π (aka policy evaluation)

- But we can fix it by only recursing a certain number of times

- This raises the a question of whether this will give us the right answer
  - We'll get to that later

```
V(π, s, k) {
  if (k == 0) return 0;
  sum = 0;
  foreach a {
    foreach s' {
      r = R[a,s,s']+ γ*V(π, s', k-1);
      sum += π(s, a)*P[a,s,s']*r;
    }
  }
  return sum;
}
```

Computing V^{π} (aka policy evaluation)

- But even this still has a massive problem
- Can you see what it is?

```
V(π, s, k) {
   if (k == 0) return 0;
   sum = 0;
   foreach a {
      foreach s’ {
          r = R[a,s,s’]+ γ*V(π, s’, k-1);
          sum += π(s, a)*P[a,s,s’]*r;
      }
   }
   return sum;
}
```

Computing V^{π} (aka policy evaluation)

It's massively **inefficient**

- V(π, s', k-1) gets recomputed once for each value of a
  - (each iteration of the outer loop)
- Worse, the recursive calls that those calls make get repeated too
- So if there are n different actions, then V(π, s', k-i) gets computed $\boldsymbol{n^i}$ **times**
- How do we fix this?

```
V(π, s, k) {
  if (k == 0) return 0;
  sum = 0;
  foreach a {
    foreach s' {
      r = R[a,s,s']+ γ*V(π, s', k-1);
      sum += π(s, a)*P[a,s,s']*r;
    }
  }
  return sum;
}
```

Dynamic programming

- Compute each value of V(π, s', k) **once only**
- Stash it in a **table**
- Use the value in the table for subsequent calls

- This is known as **top-down dynamic programming** or **memoization**
  - C.f. 214, 336, and some versions of 111

```
V(π, s, k) {
   if (k == 0) return 0;
   if (table[s,k] filled in)
     return table[s,k]
  sum = 0;
  foreach a {
    foreach s' {
       r = R[a,s,s']+ γ*V(π, s', k-1);
       sum += π(s, a)*P[a,s,s']*r;
    }
  }
  table[s,k] = sum;
  return sum;
}
```

Dynamic programming

- However, since we know we'll end up computing all the entries in table[s,k] anyway, why bother with the annoying recursion?
  - Just compute all the entries for table[s,0]
  - Then compute all the entries for table[s,1]
  - Then compute all the entries for table[s, 2]
  - Etc.

```
V(π, s, k) {
   if (k == 0) return 0;
   if (table[s,k] filled in)
     return table[s,k]
  sum = 0;
  foreach a {
    foreach s' {
        r = R[a,s,s']+ γ*V(π, s', k-1);
        sum += π(s, a)*P[a,s,s']*r;
    }
  }
  table[s,k] = sum;
  return sum;
}
```

Dynamic programming

- Here's the code

- Just call this, and then the estimated values of V are all in table[s, k]

- This is known as **bottom-up dynamic programming**

```
FillTable() {
    foreach s
        table[s,0]=0
    for i=1 to k
        foreach s {
           sum = 0;
           foreach a {
               foreach s' {
                  r = R[a,s,s']+ γ*V(π, s', k-1);
                  sum += π(s, a)*P[a,s,s']*r;
               }
             table[s,k] = sum;
        }
}
```

Dynamic programming

- Dynamic programming was originally invented by Bellman for solving MDPs

- It was called dynamic programming because
  - Programming in those days meant optimization
  - He solved an optimization involving time
  - He thought the word dynamic made it sound more impressive (no, really!)

```
FillTable() {
    foreach s
        table[s,0]=0
    for i=1 to k
        foreach s {
           sum = 0;
           foreach a {
               foreach s' {
                  r = R[a,s,s']+ γ*V(π, s', k-1);
                  sum += π(s, a)*P[a,s,s']*r;
               }
          table[s,k] = sum;
        }
}
```

Getting back to the equations…

- We're trying to compute

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}^{a}_{ss'} \left[\mathcal{R}^{a}_{ss'} + \gamma V^{\pi}(s') \right]$$

- And we basically said we could compute it by computing

$$V^{\pi}_{k}(s) = \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}^{a}_{ss'} \left[\mathcal{R}^{a}_{ss'} + \gamma V^{\pi}_{k-1}(s') \right]$$

- For large values of k
- This was Bellman's original formulation

Finding the optimal policy

- The **optimal state value function** would be the one that does whatever the best policies do in any given state:

$$V^*(s) = \max_\pi V^\pi(s)$$

- If we knew what V^* was, we could compute an **optimal action-state value function** for it:

$$Q^*(s,a) = Q^{\pi^*}(s,a) = \sum_{s'} \mathcal{P}_{ss'}^a \left[\mathcal{R}_{ss'}^a + \gamma V^*(s')\right]$$

- And back-solve the **optimal policy** from that:

$$\pi^*(s,a) = \begin{cases} 1, & a = \max_{a'} Q^*(s,a') \\ 0, & \text{otherwise} \end{cases}$$

Bellman's optimality criteria

- Bellman showed that the **optimal value function** is one that does what the **optimal policy** does for any given state:

$$V^*(s) = \max_a Q^{\pi^*}(s,a)$$

$$= \max_a \sum_{s'} \mathcal{P}^a_{ss'}\left[\mathcal{R}^a_{ss'} + \gamma V^*(s')\right]$$

Value iteration

- So we can **approximate V^*** using the same dynamic programming trick used for policy evaluation:

$$V^*(s) = \lim_{k \to \infty} V_k(s)$$

$$V_k(s) = \max_a \sum_{s'} \mathcal{P}_{ss'}^a \left[\mathcal{R}_{ss'}^a + \gamma V_{k-1}(s') \right]$$

Value iteration

Initialize V arbitrarily, e.g., $V(s) = 0$, for all $s \in \mathcal{S}^+$

Repeat

$\Delta \leftarrow 0$

For each $s \in \mathcal{S}$:

a

$v \leftarrow V(s)$

$V(s) \leftarrow \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V(s')\right]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, π, such that

$\pi(s) = \arg\max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V(s')\right]$

Conclusion

- Value iteration gives us a **greedy** policy provided we have a **perfect model** of the world
  - In the form of $P_{ss'}^{a}$ and $R_{ss'}^{a}$
- Next we'll look at **learning policies from experience** without assuming a prior model

Recall

- Optimal "value function" :

$$V^*(s) = \lim_{k\to\infty} V_k(s)$$

$$V_k(s) = \max_a \sum_{s'} \mathcal{P}^a_{ss'} \left[\mathcal{R}^a_{ss'} + \gamma V_{k-1}(s') \right]$$

Learning from Experience

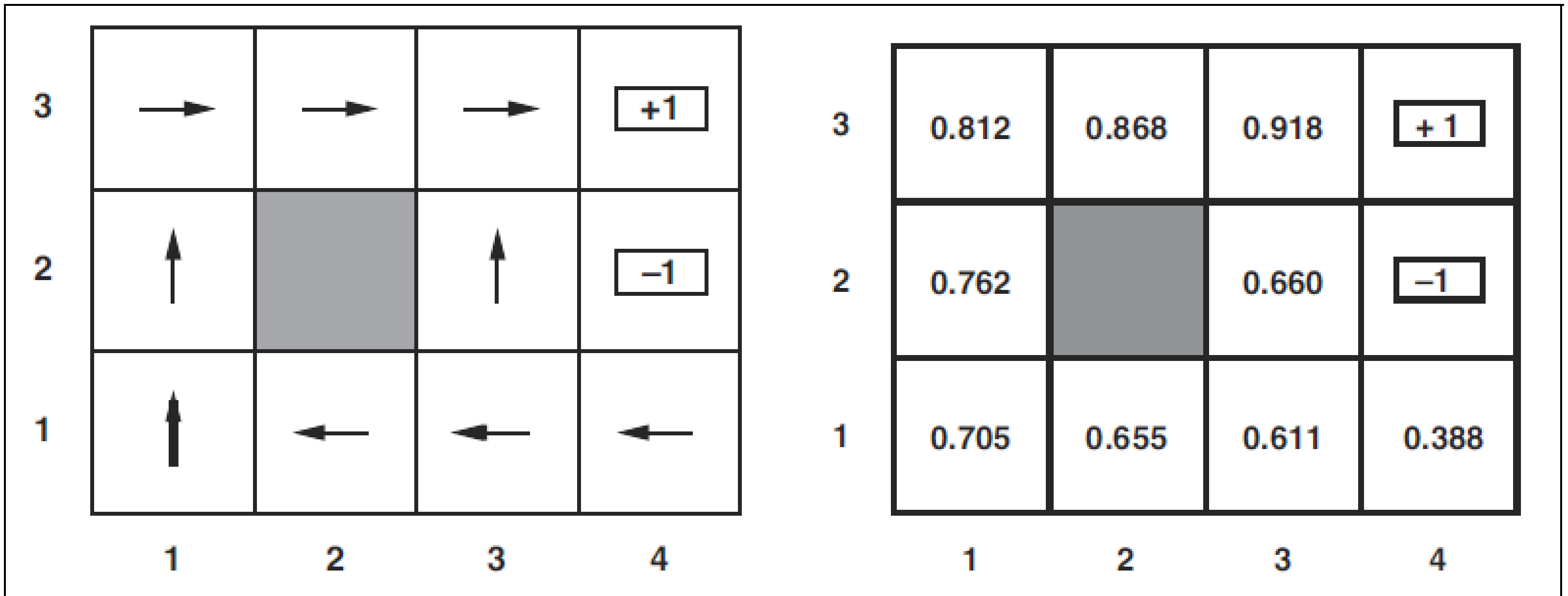
- We need
  - Model of the world $\mathcal{P}_{ss'}^{a}$,
  - Reward model $\mathcal{R}_{ss'}^{a}$,

- How do we get them?
  - One option, we write them down
    - Design reward function, physical model, etc.
  - What about uncertain environments? => **LEARN**

Gee, it's easy

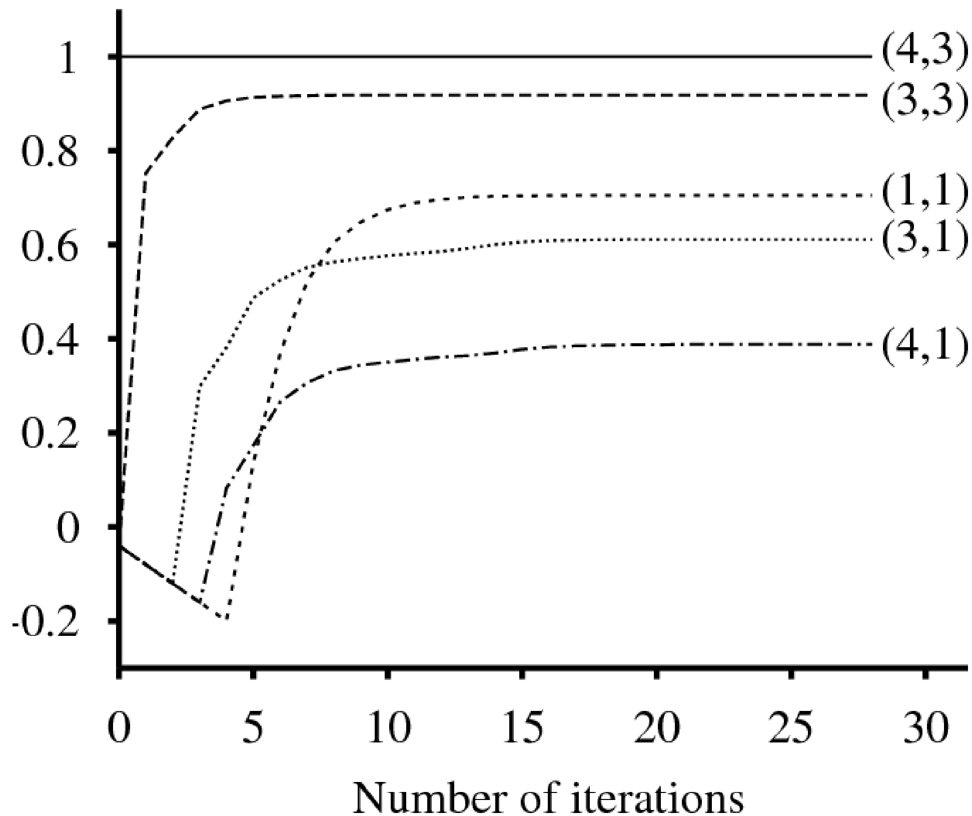
- Collect experience by moving through the world
  - $s_0, a_0, r_1, s_1, a_1, r_2, s_2, a_2, r_3, s_3, a_3, r_4, s_4, a_4, r_5, s_5, \ldots$
- Use these to estimate world, reward models
- Solve for the optimal value function
- Compute the optimal policy from it

Example



(a) (b)

From Russell and Norvig



(4,3)
(3,3)
(1,1)
(3,1)
(4,1)
1
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
30
Number of iterations

What’s wrong with that?

- Intractable for all but the simplest problems
- Spends a ton of time in low-value states

Let's start with tractability

- Do we really have to learn $\mathcal{P}_{ss'}^{a}$?

- Related question – you may be able to play Pac-Man. Does that mean you've computed the stochastic model underlying Pac-Man?
  - No.

- Idea: learn what to do next, *without* world model

TD(0)-Learning Algorithm

- Input – a **fixed** policy π to evaluate
- Initialize $V^{\pi}(s)$ to 0
- For each 'episode' (episode = series of actions)
  - Repeat until out of actions:
    1. Observe state s
    2. Perform action according to the policy $\pi(s)$
    3. $V(s) \leftarrow (1-\alpha)V(s) + \alpha[r + \gamma V(s')]$
    4. $s \leftarrow s'$

r = reward
α= learning rate
γ= discount factor

Note: this formulation is from Sutton & Barto's "Reinforcement Learning"

TD(0)-Learning

- TD(0)'s V(s) estimate will converge to $V^{\pi}(s)$
  - After an infinite number of experiences
  - If we decay the learning rate s.t.:

$$\sum_{t=0}^{\infty} \alpha_t = \infty \qquad \sum_{t=0}^{\infty} {\alpha_t}^2 < \infty$$

  - …so $\alpha_t = \dfrac{c}{c+t}$ will work

- =>We can get $V^{\pi}(s)$ more tractably… but $V^{*}(s)$?
  - And we're still spending lots of time in low-val states

Exploration vs. Exploitation

- We want to pick good actions most of the time, but also do some exploration
- Exploring means we can learn better policies
- But, we want to balance known good actions with exploratory ones
- This is called the **exploration/exploitation** problem

Let’s Explore! And exploit

- On-policy algorithms
  - Final policy is influenced by the exploration policy
  - Generally, the exploration policy needs to be “close” to the final policy
  - Can get stuck in local maxima

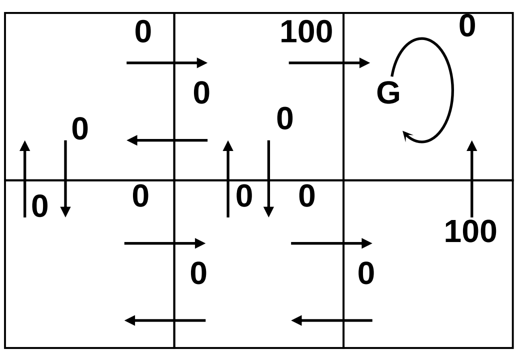
- Off-policy algorithms

  Given enough experience

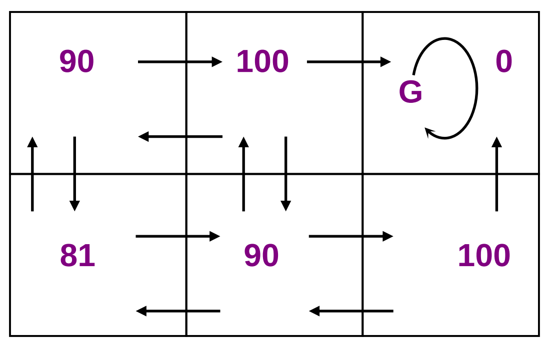
  - Final policy is independent of exploration policy
  - Can use arbitrary exploration policies
  - Will not get stuck in local maxima

We’ll learn Q

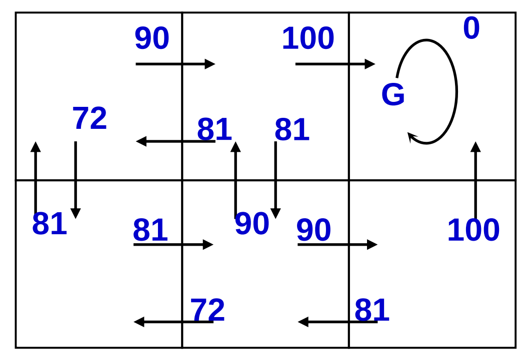
- Rather than V*(s), we’ll learn:
  - Q(s, a) = the expected utility of taking a particular action **a** in state **s**



***r*(*state*, *action*)**
immediate reward values



V^*(*state*) values



***Q*(*state*, *action*) values**

Picking Actions

ε-greedy
  – Pick best (greedy) action with probability ε
  – Otherwise, pick a random action
- Boltzmann (Soft-Max)
  – Pick an action based on its Q-value

$$P(a|s) = \frac{e^{\left(\frac{Q(s,a)}{\tau}\right)}}{\sum_{a'} e^{\left(\frac{Q(s,a')}{\tau}\right)}}$$

…where τ is the “temperature”

Two methods

- SARSA (on-policy)
- Q-learning (off-policy)

SARSA

- SARSA iteratively approximates the state-action value function, Q
  - SARSA learns the policy and the value function simultaneously
- Keep an estimate of Q(s, a) in a table
  - Update these estimates based on experiences
  - Estimates depend on the exploration policy
  - SARSA is an on-policy method
  - Policy is derived from current value estimates

SARSA Algorithm

1. Initialize $Q(s, a)$ to small random values, $\forall s, a$
2. Observe state, s
3. $a \leftarrow \pi(s)$
   (policy derived from Q, e.g. ε-greedy)
4. Observe next state, s', and reward, r
5. $Q(s, a) \leftarrow (1-\alpha)Q(s, a) + \alpha(r + \gamma Q(s', \pi(s')))$
6. Go to 2

- $0 \le \alpha \le 1$ is the learning rate
  - We should decay this, just like TD

Q-Learning [Watkins & Dayan, 92]

- Q-learning iteratively approximates the state-action value function, Q
  - Like SARSA, we won't estimate a world model
  - Learns the value function and policy simultaneously
- Keep an estimate of Q(s, a) in a table
  - Update these estimates as we gather more experience
  - Estimates **do not** depend on exploration policy
  - Q-learning is an **off-policy** method

Q-Learning Algorithm

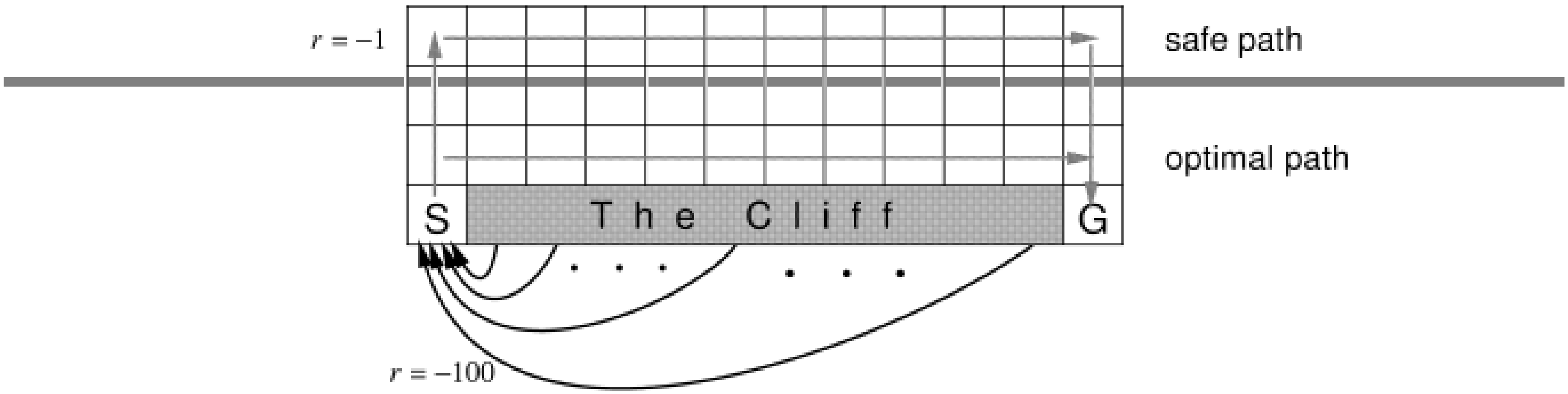
1. Initialize $Q(s, a)$ to small random values, $\forall s, a$
   (what if you make them 0? What if they are big?)
2. Observe state, s
3. Pick action a using policy derived from Q
4. Observe next state, s', and reward, r
5. $Q(s, a) \leftarrow (1 - \alpha)Q(s, a) + \alpha(r + \gamma \max_{a'} Q(s', a'))$
6. $s \leftarrow s'$
7. Go to 2

$0 \leq \alpha \leq 1$ is the learning rate & we should decay α, just like in TD

This formulation is from Sutton & Barto's "Reinforcement Learning"

Q-learning vs. SARSA

- SARSA:
  - $Q(s, a) \leftarrow (1-\alpha)Q(s, a) + \alpha(r + \gamma Q(s', \pi(s')))$
- Q-learning:
  - $Q(s, a) \leftarrow (1 - \alpha)Q(s, a) + \alpha(r + \gamma \max_{a'} Q(s', a'))$
- In both algorithms, actions chosen according to the Q being learned (exploit while exploring)…
  - So why is Q-learning "off-policy" ?


r = −1
safe path
optimal path
S
The Cliff
G
r = −100


Sarsa
Q-learning
Reward per epsiode
−25
−50
−75
−100
0
100
200
300
400
500
Episodes

Reinforcement Learning for Robotics?

- Challenges
  - Actions have physical consequences
  - State-action space is continuous/high-dim
    - Sparse! And, how to get $\max_{a'}()$?
  - Bottom line: RL not feasible in robots w/out modifications
- Good news
  - Good framework to start with
  - Parallel to human/animal learning
    - (vs. input/output pairs in supervised learning)
  - Modifications have been developed to port RL to robots